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# Branes from a non-Abelian $(2,0)$ tensor multiplet with 3-algebra

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## Abstract

In this paper, we study the equations of motion for non-Abelian  $\mathcal{N} = (2, 0)$  tensor multiplets in six dimensions, which were recently proposed by Lambert and Papageorgakis. Some equations are regarded as constraint equations. We employ a loop extension of the Lorentzian three-algebra (3-algebra) and examine the equations of motion around various solutions of the constraint equations. The resultant equations take forms that allow Lagrangian descriptions. We find various  $(5 + d)$ -dimensional Lagrangians and investigate the relation between them from the viewpoint of M-theory duality.

# 1 Introduction

Since its discovery, M-theory has been intensively studied from various viewpoints, such as string duality and its applications to supersymmetric gauge theories. Despite extensive study since the 1990s, the basic properties of M-theory, including its fundamental degrees of freedom, still remain mysterious. However, there are a number of aspects of M-theory which have been clarified. For instance, the low-energy limit of this theory is 11-dimensional supergravity and, at least in the long-wavelength approximation, it accommodates two kinds of extended object, M-theory two-branes (M2-branes) and five-branes (M5-branes), which couple to the three-form gauge fields in 11-dimensional supergravity.

Similarly, there are still numerous aspects of M-branes to unveil. In particular, the world-volume description of multiple M-branes is interesting in the context of the AdS/CFT correspondence and, more importantly, M-branes are believed to be described by novel interacting superconformal field theories, such as a three-dimensional  $\mathcal{N} = 8$  superconformal field theory for M2-branes and a six-dimensional superconformal field theory of  $(2, 0)$  tensor multiplets for M5-branes. However, the formulation of the theory at the fundamental level poses a difficult problem which has remained unresolved for a long time.

Recently, a world-volume description of multiple M2-branes using a new kind of symmetry structure, the so-called Lie 3-algebra, was proposed independently by Bagger and Lambert [1, 2], and Gustavsson [3] (BLG). Since then, significant progress has been made in understanding the BLG theory and 3-algebra itself. In the course of the study of 3-algebra, it was first conjectured [4] and later proved<sup>1</sup> [6, 7] that the only finite-dimensional Lie 3-algebras with a positive definite metric are the trivial one,  $\mathcal{A}_4$ , and the direct sum of these algebras. Multiple M2-branes can also be reformulated in the context of the double Chern-Simons theory with the usual Lie group symmetry [8, 9]<sup>2</sup>, and thus the 3-algebra might not be indispensable in describing multiple M2-branes. On the contrary, an  $\mathcal{N} = 6$  Chern-Simons-matter theory in three dimensions, the so-called ABJM theory, also revealed hidden 3-algebraic structures [11], and S-matrix analysis of three-dimensional (3D)  $\mathcal{N} = 8$  on-shell supermultiplets suggested that interaction has 3-algebraic structures [12]. Therefore, 3-algebra could play a crucial role in the analysis of the dynamics of M-theory, but a better understanding of its attributes is necessary. Moreover, since we can formulate a system of an infinite number of M2-branes which can be condensed

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<sup>1</sup>We have been informed of the paper [5] which claims to provide the first proof to this conjecture.

<sup>2</sup>There is recent research on the hidden maximal supersymmetry of the Chern-Simons-matter theory with less than  $\mathcal{N} = 8$  supersymmetry; for example, see [10].

to a single M5-brane by means of infinite-dimensional Lie 3-algebra<sup>3</sup>, it would also be possible to use 3-algebra to formulate a system of interacting M5-branes manifesting (2, 0) supersymmetry.

Recently, Lambert and Papageorgakis [17] proposed a set of on-shell supersymmetry transformations for non-Abelian (2, 0) tensor multiplets in six dimensions, using a 3-algebraic structure. This approach immediately invokes the following question: to what extent is it related to multiple M5-branes systems?

The supersymmetry transformations proposed by Lambert and Papageorgakis were as follows:

$$\begin{aligned}
\delta X_A^I &= i\bar{\epsilon}\Gamma^I\Psi_A, \\
\delta\Psi_A &= \Gamma^\mu\Gamma^I D_\mu X_A^I\epsilon + \frac{1}{3!}\frac{1}{2}\Gamma_{\mu\nu\lambda}H_A^{\mu\nu\lambda}\epsilon - \frac{1}{2}\Gamma_\lambda\Gamma^{IJ}C_B^\lambda X_C^I X_D^J f^{CDB}{}_A\epsilon, \\
\delta H_{\mu\nu\lambda A} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu}D_{\lambda]}\Psi_A + i\bar{\epsilon}\Gamma^I\Gamma_{\mu\nu\lambda\kappa}C_B^\kappa X_C^I\Psi_D f^{CDB}{}_A, \\
\delta\tilde{A}_\mu^B{}_A &= i\bar{\epsilon}\Gamma_{\mu\lambda}C_C^\lambda\Psi_D f^{CDB}{}_A, \\
\delta C_A^\mu &= 0.
\end{aligned} \tag{1.1}$$

The three-form  $H_{\mu\nu\lambda A}$  satisfies the linear self-duality condition:

$$H_{\mu\nu\lambda A} = \frac{1}{3!}\epsilon_{\mu\nu\lambda\tau\sigma\rho}H^{\tau\sigma\rho}{}_A. \tag{1.2}$$

The scalar fields  $X^I$ , the fermions  $\Psi$ , and the self-dual field  $H_{\mu\nu\rho}$  form a (2, 0) tensor multiplet in six dimensions. The two world-volume vectors  $\tilde{A}_\mu^B{}_A$  and  $C_A^\mu$  are new and play an important role in the introduction of a non-Abelian structure for the tensor multiplets. The gauge covariant derivative  $D_\mu$  is defined by  $D_\mu X_A^I = \partial_\mu X_A^I - \tilde{A}_\mu^B{}_A X_B^I$ . (See Appendix A.2 and A.3 for more details.) The Greek indices such as  $\mu, \nu$ , represent the world-volume directions and run from 0 to 5.  $I, J$  indices are for the transverse directions to the world-volume,  $I, J = 6, \dots, 10$ .  $A, B, \dots$  denote the gauge indices of the 3-algebra symmetry.

In order for the above transformations to result in an on-shell supersymmetry on the fields, the equations of motion and the constraints of the fields are necessary for their closure. These equations were derived in [17] and are<sup>4</sup>:

$$0 = (D^2 X^I)_A - \frac{i}{2}\bar{\Psi}_C C_B^\nu \Gamma_\nu \Gamma^I \Psi_D f^{CDB}{}_A + C_B^\nu C_{\nu G} X_C^J X_E^J X_F^I f^{EFG}{}_D f^{CDB}{}_A, \tag{1.3}$$

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<sup>3</sup>The Lie 3-algebra resulting in a Nambu-bracket structure was studied in detail by [13, 14, 15, 16]. The approach of employing an infinite-dimensional algebra is independent from the no-go theorem mentioned above [4, 6, 7].

<sup>4</sup>We find that the sign for the last term in (1.3) should be “+” and also the sign in (1.6) is corrected to be “+.”

$$0 = (D_{[\mu}H_{\nu\lambda\rho]})_A + \frac{1}{4}\epsilon_{\mu\nu\lambda\rho\sigma\tau}C_B^\sigma X_C^I D^\tau X_D^I f^{CDB}{}_A + \frac{i}{8}\epsilon_{\mu\nu\lambda\rho\sigma\tau}C_B^\sigma \bar{\Psi}_C \Gamma^\tau \Psi_D f^{CDB}{}_A, \quad (1.4)$$

$$0 = \Gamma^\mu (D_\mu \Psi)_A + X_C^I C_B^\nu \Gamma_\nu \Gamma^I \Psi_D f^{CDB}{}_A, \quad (1.5)$$

$$0 = \tilde{F}_{\mu\nu}{}^B{}_A + C_C^\lambda H_{\mu\nu\lambda D} f^{CDB}{}_A, \quad (1.6)$$

$$0 = D_\mu C_A^\nu = C_C^\mu C_D^\nu f^{CDB}{}_A, \quad (1.7)$$

$$0 = C_C^\rho (D_\rho X^I)_D f^{CDB}{}_A = C_C^\rho (D_\rho \Psi)_D f^{CDB}{}_A = C_C^\rho (D_\rho H_{\mu\nu\lambda})_D f^{CDB}{}_A, \quad (1.8)$$

where  $\tilde{F}_{\mu\nu}{}^B{}_A = [D_\mu, D_\nu]^B{}_A$ .

At first, these equations seem suitable for describing multiple M5-branes because of their non-Abelian structure introduced by 3-algebra. However, it was pointed out in [17] that when a simple Lorentzian 3-algebra or the  $\mathcal{A}_4$ -algebra is adopted, the equations describe multiple 4-branes and not 5-branes. With the use of these algebras, one of the world-volume directions is eliminated through equation (1.8), and the resulting equations of motion can describe the five-dimensional (5D) supersymmetric Yang-Mills (SYM) theory. In other words, we ultimately describe D4-branes rather than M5-branes. This is in many aspects similar to the "M2 to D2" scenario [18, 19, 20] proposed for the BLG theory, especially for the Lorentzian case. One may therefore wonder whether the non-Abelian tensor multiplet can describe more than the dynamics of D4-branes and 5-branes of M-theory. In this paper, we address this question by examining equations (1.3)–(1.8) based on a 3-algebra of infinite dimensions.

Through a systematic study of the fundamental identity, a class of Lorentzian 3-algebras has been proposed in [21], which includes the simplest Lorentzian 3-algebra used in [17, 19] as a special case. In this paper, we use only one of the proposed 3-algebras:

$$[u^0, u^a, u^b] = 0, \quad (1.9)$$

$$[u^0, u^a, T^{(i\vec{m})}] = m^a T^{(i\vec{m})}, \quad (1.10)$$

$$[u^0, T^{(i\vec{m})}, T^{(j\vec{n})}] = m^a \delta^{ij} \delta^{\vec{m}+\vec{n}} u^a + i f^{ij}{}_k T^{(k, \vec{m}+\vec{n})}, \quad (1.11)$$

$$[u^a, T^{(i\vec{m})}, T^{(j\vec{n})}] = -m^a \delta^{ij} \delta^{\vec{m}+\vec{n}} u^0, \quad (1.12)$$

$$[T^{(i\vec{m})}, T^{(j\vec{n})}, T^{(k\vec{l})}] = -i f^{ijk} \delta^{\vec{m}+\vec{n}+\vec{l}} u^0, \quad (1.13)$$

with inner products of

$$\langle T^{(i\vec{m})}, T^{(j\vec{n})} \rangle = \delta^{ij} \delta^{\vec{m}+\vec{n}}, \quad \langle u^a, u_b \rangle = \delta_b^a, \quad \langle u^0, u_0 \rangle = 1. \quad (1.14)$$

Other combinations of inner products are all equal to zero. It should be noted that the three-bracket is defined as  $[T^A, T^B, T^C] = if^{ABC}{}_D T^D$ , and  $T^A$  as  $T^A = \{u^0, u^a, u^{\underline{0}}, u^{\underline{a}}, T^{(i\vec{m})}\}$ . Namely the index such as  $A$  represents different kinds of generator; the Euclidean part  $i, j, \dots$  accompanied with a  $d$ -dimensional numerical vector such as  $\vec{m}$ , and the center part  $0, a, \underline{0}$  and  $\underline{a}$ .  $d$  denotes an integer and the indices  $a$  and  $b$  run from 1 to  $d$ .  $u^{\underline{0}}$  and  $u^{\underline{a}}$  are center elements of the algebra. Furthermore,  $f^{ij}{}_k$  is the structure constant of a Lie algebra,  $[T^{(i\vec{m})}, T^{(j\vec{n})}] = if^{ij}{}_k T^{(k, \vec{m}+\vec{n})}$ . Lastly,  $\vec{m}$  is a  $d$ -dimensional vector whose components are  $m^a$  ( $a = 1, \dots, d$ ) and are considered to be integers. A more detailed description of this 3-algebra is provided in Appendix A.1.

Applying this infinite-dimensional 3-algebra to the BLG theory, we can obtain the SYM theory on a torus [21]. Specifically, vector  $\vec{m}$  turns out to play the role of a Kaluza-Klein (KK) momentum vector on this torus. Thanks to the KK-momentum, new world-volume directions can be introduced and the BLG theory based on the 3-algebra is able to describe more than two-dimensional branes. The U-duality relation of D-branes wrapping on the torus has been studied in [22]. We therefore expect to get not only 4-branes but also higher-dimensional-branes using the 3-algebraic field equations (1.3)–(1.8).

In the following sections, we analyze these field equations with the use of 3-algebra (1.9)–(1.14). We will see that it is important to understand the constraints of  $C^\mu$ , *i.e.*, (1.7), in order to obtain the effective action of branes from the field equations. In section 2.3, we consider the most generic case of  $C^\mu = C_0^\mu u^0 + C_a^\mu u^a + C_{(i\vec{m})}^\mu T^{(i\vec{m})}$ . In this case, we can describe the effective action of  $(4+d)$ -brane wrapping on a torus  $T^d$ . More specifically, the brane is described in target space  $\mathbf{R}^{1,5} \times T^d \times \mathbf{R}^{4-d}$ . In section 2.4, we consider the case of  $C^\mu = C_a^\mu u^a$ . For  $C_0^\mu = 0$ , we can describe the effective action of 5-branes wrapping on the torus  $T^d$ , including non-covariant massive vector bosons. A SYM type action is recovered when  $d = 1$ . Regardless of the non-Abelian structure introduced by 3-algebra, the resulting 5-branes theory turns out to be Abelian. These 5-branes are in the target space  $\mathbf{R}^{1,5-d} \times T^d \times \mathbf{R}^4$ . In section 2.5, we consider the case of  $C_A^\mu = 0$  and recover the second-order Pasti-Sorokin-Tonin (PST [23]) type action of NS5-branes [24]. The target space of the 5-branes is reduced to  $\mathbf{R}^{1,9-d} \times M_{d+1}$ , where  $M_{d+1}$  is a  $(d+1)$ -dimensional manifold. Namely, the M-theory is compactified on  $M_{d+1}$ . In section 3, we concentrate on the case of  $d = 1$  under different  $C^\mu$ , and several 5-branes obtained in the preceding sections and their relations are investigated. The latter can be identified as 5-branes in a type IIA/IIB string theory. We find that the S-dual

relation between D5-branes and NS5-branes is naturally realized. Finally, section 4 is devoted to discussion and concludes with a summary. We also comment on recent research on the 5D maximally supersymmetric Yang-Mills (MSYM) theory proposed by [25, 26], from our viewpoint. Three appendices provide a summary of notations and conventions, and supplementary discussions.

## 2 Analysis of the equations of motion

In this section, we examine the equations of motion (1.3)–(1.8), with the infinite dimensional extension of Lorentzian type 3-algebra (1.9)–(1.14). The basic properties of this 3-algebra are summarized in Appendix A.1. The equation (1.7) is considered as the constraints for  $C_A^\mu$ . We thus start with solving these constraints, and then move on to the examination of the rest.

### 2.1 The gauge fields and the constraints for $C_A^\mu$ fields

First, in this paper we assume that the gauge field  $\tilde{A}_\mu^B{}_A$  is accompanied with the structure constant,  $\tilde{A}_\mu^B{}_A \equiv A_{\mu CD} f^{CDB}{}_A$ . It should be noted that though this is a requirement for the BLG model (M2-brane case) due to Chern-Simons term, it is not necessary in this case. But we also adopt this definition here, since it guarantees that the covariant derivative acts on the three-bracket as a derivation. Because of the limited form of the structure constant, some components of the gauge field vanish;

$$\tilde{A}_\mu^0{}_{\underline{0}} = \tilde{A}_\mu^a{}_{\underline{a}} = \tilde{A}_\mu^A{}_0 = \tilde{A}_\mu^A{}_a = \tilde{A}_\mu^{\underline{0}}{}_A = \tilde{A}_\mu^{\underline{a}}{}_A = 0. \quad (2.1)$$

This fact simplifies our analysis. The nonzero components of the gauge fields are summarized in Appendix A.3.

Next, we consider the first equation of (1.7):

$$(D_\mu C^\nu)_A = 0. \quad (2.2)$$

For  $u^0$  and  $u^a$  components, this condition immediately means

$$\partial_\mu C_0^\nu = \partial_\mu C_a^\nu = 0 \quad (2.3)$$

for arbitrary  $\mu$  and  $\nu$ . Therefore,  $C_0^\mu$  and  $C_a^\mu$  are constants. As for  $u^{\underline{0}}$  and  $u^{\underline{a}}$  components, since in all the other places  $C_A^\mu$  always appear with the structure constant  $f^{BCD}{}_A$ ,  $C_{\underline{0}}^\mu$  and  $C_{\underline{a}}^\mu$  show up only in these constraint equations:

$$\partial_\mu C_{\underline{0}}^\nu = \tilde{A}_\mu^a{}_{\underline{0}} C_a^\nu + \tilde{A}_\mu^{(i\tilde{m})}{}_{\underline{0}} C_{(i\tilde{m})}^\nu,$$

$$\partial_\mu C_{\underline{a}}^\nu = \tilde{A}_\mu^0 C_0^\nu + \tilde{A}_\mu^{(i\vec{m})} C_{(i\vec{m})}^\nu. \quad (2.4)$$

Therefore,  $C_0^\nu$  and  $C_{\underline{a}}^\nu$  are completely determined by these two equations. As we will argue in section 2.2, the components associated with the center elements of the 3-algebra are regarded as the ghost fields. Therefore, these conditions imply that the ghost fields are excited by the physical fields. To avoid it, we will impose the condition that the ghost fields stay constant:

$$\partial_\mu C_{\underline{0}}^\nu = \partial_\mu C_{\underline{a}}^\nu = 0 \quad (2.5)$$

in time evolution.

There are also constraints imposed on  $C_{(i\vec{m})}^\nu$ , which are

$$0 = (\tilde{D}_\mu C^\nu)_{(i\vec{m})} - \tilde{A}_\mu^0 C_0^\nu - \tilde{A}_\mu^a C_a^\nu \quad (2.6)$$

where the covariant derivative  $\tilde{D}_\mu$  is given by (A.21):

$$(\tilde{D}_\mu C^\nu)_{(i\vec{m})} = \partial_\mu C_{(i\vec{m})}^\nu - \tilde{A}_{\mu(i\vec{m})}^{(i\vec{m})} C_{(i\vec{m})}^\nu + i[A_\mu, C^\nu]_{(i\vec{m})}. \quad (2.7)$$

Finally, we examine the second equation of (1.7):

$$[C^\mu, C^\nu, T^B]_A = 0. \quad (2.8)$$

For the various combinations of  $A, B = \{(i\vec{m}), 0, a, \underline{0}, \underline{a}\}$ , we have

$$\begin{aligned} 0 &= f^{ki} C_0^{[\mu} C_{(k, \vec{m}-\vec{n})}^{\nu]} = m^b C_0^{[\mu} C_b^{\nu]} \\ &= m^b C_b^{[\mu} C_{(i\vec{m})}^{\nu]} - [C^\mu, C^\nu]_{(i\vec{m})} = m^a C_0^{[\mu} C_{(i\vec{m})}^{\nu]} = C_{(k\vec{\ell})}^{[\mu} C_{(k, -\vec{\ell})}^{\nu]} \end{aligned} \quad (2.9)$$

where in the first equation  $\vec{m} \neq \vec{n}$ , and the repeated indices in a single term (here denoted as  $b, k$  and the vector  $\vec{\ell}$ ) are all implicitly summed over, and we will use this contraction rule throughout this paper.  $f^{ij}_k$  is the structure constant for a conventional Lie algebra which is embedded into our 3-algebra, and

$$[\phi, \varphi]_{(i\vec{m})} \equiv i f^{jk}_i \phi_{(j\vec{n})} \varphi_{(k, \vec{m}-\vec{n})}. \quad (2.10)$$

In (2.9), the last condition is trivial. If only one direction of  $C^\mu$  is non-vanishing, all the constraints are trivially satisfied. Otherwise, these constraints give restriction on  $C$  fields. To solve the constraints, we consider the following possibilities:

1. All  $C_0^\mu$ ,  $C_a^\mu$  and  $C_{(i\vec{m})}^\mu$  are nonzero:

In this most generic case, to satisfy the constraints there have to be the relations;  $C_a^\mu \propto C_0^\mu$  and  $C_{(i\vec{m})}^\mu \propto C_0^\mu$ . Therefore, we have the following conditions,

$$C_a^\mu = v_a C_0^\mu, \quad C_{(i\vec{m})}^\mu = v_{(i\vec{m})} C_0^\mu, \quad (2.11)$$

where  $v_a$  and  $v_{(i\vec{m})}$  are chosen in common for all  $\mu$ . Note that  $v_{(i\vec{m})}$  are commuting each other with respect to the commutator (2.10),  $[v, v]_{(i\vec{m})} = 0$ .

2.  $C_0^\mu \neq 0$  but  $C_a^\mu = C_{(i\vec{m})}^\mu = 0$ :

This case is included in the previous case with  $v_a = v_{(i\vec{m})} = 0$ .

3.  $C_0^\mu = C_{(i\vec{m})}^\mu = 0$  and  $C_a^\mu \neq 0$ :

In this case,  $C_a^\mu$  would take arbitrary constant values which are not necessarily proportional to one another.

4. All  $C_0^\mu = C_a^\mu = C_{(i\vec{m})}^\mu = 0$  case:

In this case, the non-Abelian interactions are almost turned off.

We do not try to exhaust all the possibilities but look at interesting cases. In the following subsections, we are going to investigate the equations of motion for each of the above cases. (The case 2 is included in the case 1.) Before going to the analysis, we discuss the decoupling of the ghost fields.

## 2.2 Decoupling of the ghost fields

Since we do not start with a Lagrangian but the equations of motion, the existence of the negative component of the generators does not immediately mean the existence of the negative norm states (*i.e.*, fields with wrong sign kinetic terms). However, we aim to construct effective actions with respect to various values of  $C_A^\mu$  and it is plausible that we can introduce a prescription to deal with the ghost fields. In this paper, we take the strategy used in [19, 28, 29], where the shift symmetry existing for the center components are gauged and these components are gauged away. At the same time, the equations of motion for newly introduced gauge fields provide constraints for the paired components, and they become non-dynamical.

As we will see,  $X^I$  and  $\Psi$  have an ordinary Lagrangian description even for the non-Abelian case. For the self-dual three-form field  $H_{\mu\nu\rho}$ , the treatment is slightly different for each situation. We will give a sketch of the decoupling mechanism here and will supplement comments later in each subsection.



Since the interaction terms always involve the structure constant  $f^{BCD}{}_A$ ,  $X_{\underline{0}}^I$ ,  $X_{\underline{a}}^I$ ,  $\Psi_{\underline{0}}$  and  $\Psi_{\underline{a}}$  components appear in the action only through the kinetic terms:

$$\begin{aligned}\mathcal{L}_{gh} &= - (D_\mu X^I)_0 (D^\mu X^I)_{\underline{0}} - (D_\mu X^I)_a (D^\mu X^I)_{\underline{a}} \\ &\quad + \frac{i}{2} (\bar{\Psi}_0 \Gamma^\mu D_\mu \Psi_{\underline{0}} + \bar{\Psi}_a \Gamma^\mu D_\mu \Psi_{\underline{a}}) \\ &= - (D_\mu X^I)_\alpha (D^\mu X^I)_{\underline{\alpha}} + \frac{i}{2} \bar{\Psi}_\alpha \Gamma^\mu D_\mu \Psi_{\underline{\alpha}},\end{aligned}\tag{2.12}$$

where  $\alpha = (0, a)$  and  $a = 1, \dots, d$ . It should be noted that this kinetic term correctly reproduces the kinetic term of the  $u^{\underline{a}}$  part. Because of the restricted form of the gauge field  $\tilde{A}_\mu$ , the covariant derivative has to take the following form:

$$(D_\mu \phi)_{\underline{\alpha}} = \partial_\mu \phi_{\underline{\alpha}} + (\text{terms not including } \phi_{\underline{\alpha}}),\tag{2.13}$$

where  $\phi_{\underline{\alpha}}$  means  $X_{\underline{\alpha}}^I$  or  $\Psi_{\underline{\alpha}}$ . Therefore, there exist the following shift symmetries:

$$X_{\underline{\alpha}}^I \rightarrow X_{\underline{\alpha}}^I + \xi_{\underline{\alpha}}^I, \quad \Psi_{\underline{\alpha}} \rightarrow \Psi_{\underline{\alpha}} + \eta_{\underline{\alpha}},\tag{2.14}$$

with constant  $\xi_{\underline{\alpha}}^I$  and  $\eta_{\underline{\alpha}}$ . We now promote these shift symmetries to gauged ones (space-time dependent):

$$\xi_{\underline{\alpha}}^I \rightarrow \xi_{\underline{\alpha}}^I(x), \quad \eta_{\underline{\alpha}} \rightarrow \eta_{\underline{\alpha}}(x),\tag{2.15}$$

by introducing gauge fields:

$$\partial_\mu X_{\underline{\alpha}}^I \rightarrow \partial_\mu X_{\underline{\alpha}}^I + a_{\mu \underline{\alpha}}^I, \quad \partial_\mu \Psi_{\underline{\alpha}} \rightarrow \partial_\mu \Psi_{\underline{\alpha}} + b_{\mu \underline{\alpha}},\tag{2.16}$$

that obey the transformation law:

$$a_{\mu \underline{\alpha}}^I \rightarrow a_{\mu \underline{\alpha}}^I - \partial_\mu \xi_{\underline{\alpha}}^I(x), \quad b_{\mu \underline{\alpha}} \rightarrow b_{\mu \underline{\alpha}} - \partial_\mu \eta_{\underline{\alpha}}(x).\tag{2.17}$$

Now we can gauge away  $X_{\underline{\alpha}}^I$  and  $\Psi_{\underline{\alpha}}$  by a gauge choice, and the equations of motion of these gauge fields impose the constraints:

$$\partial_\mu X_\alpha^I = \Psi_\alpha = 0,\tag{2.18}$$

on the conjugate fields  $X_\alpha^I$  and  $\Psi_\alpha$ , which used to satisfy the free equations of motion:

$$\partial^\mu \partial_\mu X_\alpha^I = \Gamma^\mu \partial_\mu \Psi_\alpha = 0.\tag{2.19}$$

In this way, we can eliminate the unwanted fields, and instead obtain some “moduli” fields. In the following analysis of the equations of motion, we will assume that these ghost fields have already been eliminated and

$$X_0^I = \lambda_0^I, \quad X_a^I = \lambda_a^I, \quad \Psi_0 = \Psi_a = 0,\tag{2.20}$$

are imposed, where  $\lambda_0^I$  and  $\lambda_a^I$  are certain constant vectors and will be identified as moduli of the theory.

For the three-form field  $H_{\mu\nu\rho}$ , the situation is more complicated since it does not allow a simple Lagrangian description due to self-duality. However, we can still observe a shift symmetry:

$$H_{\mu\nu\rho\,\underline{\alpha}} \rightarrow H_{\mu\nu\rho\,\underline{\alpha}} + \zeta_{\mu\nu\rho\,\underline{\alpha}}, \quad (2.21)$$

in the equations of motion for  $H_{\mu\nu\rho\,\underline{\alpha}}$ , and then by gauging it we can eliminate  $H_{\mu\nu\rho\,\underline{\alpha}}$ . To do so, we introduce a gauge field that transforms as

$$G_{\mu\nu\rho\,\underline{\alpha}} \rightarrow G_{\mu\nu\rho\,\underline{\alpha}} - \zeta_{\nu\rho\sigma\,\underline{\alpha}}. \quad (2.22)$$

If the kinetic term of  $H$  field were like:

$$\frac{1}{2} \frac{1}{3!} H_{\alpha}^{\mu\nu\rho} (H_{\mu\nu\rho\,\underline{\alpha}} + G_{\mu\nu\rho\,\underline{\alpha}}), \quad (2.23)$$

the gauge field equation of motion would lead to the constraints:

$$H_{\mu\nu\rho\,0} = H_{\mu\nu\rho\,a} = 0. \quad (2.24)$$

However, because of self-duality, it does not go easily. We will discuss the treatment of  $H_{\mu\nu\rho\,0}$  and  $H_{\mu\nu\rho\,a}$  in each case separately.

## 2.3 Generic case

**$C$  fields and gauge fields** In this subsection, we consider the case with all  $C_A^\mu$  being nonzero,  $C^\mu = C_0^\mu u^0 + C_a^\mu u^a + C_{(i\vec{m})}^\mu T^{(i\vec{m})}$ . This is the case 1 (including the case 2) of the classification in section 2.1. As seen in section 2.1,  $C_a^\mu$  and  $C_{(i\vec{m})}^\mu$  components are proportional to  $C_0^\mu$  component:

$$C_a^\mu = v_a C_0^\mu, \quad C_{(i\vec{m})}^\mu = v_{(i\vec{m})} C_0^\mu. \quad (2.25)$$

It should be noted that  $C_0^\mu$  are constant due to the equation of motion. Thus it is always possible, by Lorentz rotation, to align  $C_0^\mu$  into one direction, say  $\mu = \tilde{\mu}$ , and to make the other components vanish,  $C_0^{\mu \neq \tilde{\mu}} = 0$ . Therefore, in this case, we assume that the auxiliary field  $C_0^\mu$  takes nonzero value only for  $\mu = \tilde{\mu}$  without loss of generality. We also assume that  $C_{\underline{0}}^\mu$  and  $C_{\underline{a}}^\mu$  components are not coupled to the other physical fields,  $\partial_\nu C_{\underline{0}}^\mu = \partial_\nu C_{\underline{a}}^\mu = 0$ . This condition, together with (2.4), implies

$$v_a \tilde{A}_{\mu\,\underline{0}}^a = -v_{(i\vec{m})} \tilde{A}_{\mu}^{(i\vec{m})}{}_{\underline{0}}, \quad \tilde{A}_{\mu\,\underline{a}}^0 = -v_{(i\vec{m})} \tilde{A}_{\mu}^{(i\vec{m})}{}_{\underline{a}}, \quad (2.26)$$

for non-zero  $C_0^\mu$ . The anti-symmetry  $\tilde{A}_\mu^0 = -\tilde{A}_\mu^a$  gives the relation

$$v_{(i,-\vec{m})} \left( \tilde{A}_\mu^0 + v_a \tilde{A}_\mu^a \right) = 0. \quad (2.27)$$

Now we consider the constant values of the  $C_{0,a}$  fields,  $C_0^\mu$  and  $v_a$ , as moduli of the effective theory. Then  $v_{(i\vec{m})}$  is also preferred to be taken as an unrestricted parameter here. This requires the condition  $\tilde{A}_\mu^0 = -v_a \tilde{A}_\mu^a$ . Note that though this  $v_{(i\vec{m})}$  turns out to be zero as a result of the gauge field equation of motion, we leave  $v_{(i\vec{m})}$  unrestricted for a while. On such  $v_{(i\vec{m})}$ , the condition  $(\tilde{D}_\mu v)_{(i\vec{m})} = 0$  is imposed due to (2.6).

By employing the relationship  $\tilde{A}_\mu^B = A_{\mu CD} f^{CDB}_A$  and  $\tilde{A}_\mu^a = im^a A_{\mu 0(i\vec{m})}$ , together with the above relationships such as  $\tilde{A}_\mu^0 = -v_a \tilde{A}_\mu^a$ , the gauge fields are represented by the ones without tilde as

$$\tilde{A}_\mu^a = im^a A_{\mu(i\vec{m})}, \quad (2.28)$$

$$\tilde{A}_\mu^0 = -im^a v_a A_{\mu(i\vec{m})}, \quad (2.29)$$

$$\tilde{A}_\mu^0 = im^a v_{(i,-\vec{m})} A_{\mu(i\vec{m})}, \quad (2.30)$$

$$\tilde{A}_\mu^{(i\vec{m})}_{(j\vec{n})} = f^{ki}_j A_{\mu(k,\vec{n}-\vec{m})}, \quad (2.31)$$

$$\tilde{A}_\mu^{(i\vec{m})}_{(i\vec{m})} = im^a a_{\mu a}, \quad (2.32)$$

where we have defined  $A_{\mu 0(i\vec{m})} \equiv A_{\mu(i\vec{m})}$  and  $A_{\mu a0} \equiv a_{\mu a}$ . So the gauge fields are all represented by these two gauge fields. From (A.15), this relation implies

$$f^{jk}_i A_{\mu(j\vec{n})(k,\vec{m}-\vec{n})} = 0, \quad A_{\mu a(i\vec{m})} = v_a A_{\mu(i\vec{m})}. \quad (2.33)$$

From  $(D_\mu C^\rho)_A = 0$ , there are also the constraints on the field strength  $\tilde{F}_{\mu\nu}$  as

$$0 = ([D_\mu, D_\nu] C^\rho)_A = \tilde{F}_{\mu\nu}^B C_B^\rho, \quad (2.34)$$

for any gauge indices  $A$ . For each gauge index, the equation (2.34) is written as

$$0 = \left( \tilde{F}_{\mu\nu}^0 + v_{(i\vec{m})} \tilde{F}_{\mu\nu}^{(i\vec{m})} \right) C_0^\rho, \quad (2.35)$$

$$0 = v_{(i\vec{m})} \left( \tilde{F}_{\mu\nu}^0 + v_a \tilde{F}_{\mu\nu}^a \right) C_0^\rho, \quad (2.36)$$

$$0 = \left( \tilde{F}_{\mu\nu}^{(i\vec{m})} v_{(i\vec{m})} + \tilde{F}_{\mu\nu}^{(j\vec{n})} v_{(j\vec{n})} \right) C_0^\rho. \quad (2.37)$$

Since there is one non-vanishing  $C_0^\mu$ , we have the relations between the field strengths:

$$\tilde{F}_{\mu\nu}^0 = -v_{(i,-\vec{m})} \tilde{F}_{\mu\nu}^a, \quad \tilde{F}_{\mu\nu}^0 = -v_a \tilde{F}_{\mu\nu}^a,$$

$$v_{(i\vec{m})} \tilde{F}_{\mu\nu}^{(i\vec{m})} = -v_{(j\vec{n})} \tilde{F}_{\mu\nu}^{(j\vec{n})} \quad (2.38)$$

The diagonal part can be written in terms of only  $a_{\mu a}$ ,

$$\tilde{F}_{\mu\nu}^{(i\vec{m})} = -im^a f_{\mu\nu a}, \quad f_{\mu\nu a} \equiv \partial_\mu a_{\nu a} - \partial_\nu a_{\mu a}. \quad (2.39)$$

It should be noted that for the second equation of (2.38), we use the fact that  $v_{(i\vec{m})}$  is unconstrained, but by using the explicit form of the gauge fields (2.28)–(2.32) this can also be confirmed.

Analogously, the following identity:

$$0 = (D^\mu [D_\mu, D_\nu] C^\rho)_A = \left( D^\mu \tilde{F}_{\mu\nu} \right)^B{}_A C_B^\rho, \quad (2.40)$$

provides the same relations among the covariant derivatives of the field strength:

$$\left( D^\mu \tilde{F}_{\mu\nu} \right)^0{}_{\underline{a}} = v_{(i, -\vec{m})} \left( D^\mu \tilde{F}_{\mu\nu} \right)^a{}_{(i\vec{m})}, \quad \left( D^\mu \tilde{F}_{\mu\nu} \right)^0{}_{(i\vec{m})} = -v_a \left( D^\mu \tilde{F}_{\mu\nu} \right)^a{}_{(i\vec{m})}, \quad (2.41)$$

$$v_{(i\vec{m})} \left( D^\mu \tilde{F}_{\mu\nu} \right)^{(i\vec{m})}{}_{(i\vec{m})} = -v_{(j\vec{n})} \left( D^\mu \tilde{F}_{\mu\nu} \right)^{(j\vec{n})}{}_{(i\vec{m})}, \quad (2.42)$$

and we will use these relations later to simplify the gauge field equations of motion. Again, the diagonal part becomes

$$\left( D^\mu \tilde{F}_{\mu\nu} \right)^{(i\vec{m})}{}_{(i\vec{m})} = -im^a \partial^\mu f_{\mu\nu a}. \quad (2.43)$$

Next, we consider the equations (1.8):

$$[C^\mu, D_\mu \phi, T^B]_A = 0, \quad (2.44)$$

where  $\phi$  denotes any of  $X^I$ ,  $\Psi$  or  $H_{\mu\nu\rho}$ . From (1.3)–(1.5), (2.20) and (2.24), it is easy to see that

$$(D_\mu \phi)_0 = (D_\mu \phi)_a = 0, \quad (2.45)$$

and thus  $u^0$  and  $u^a$  components do not enter this constraint equation.  $(D_\mu \phi)_{\underline{0}}$  and  $(D_\mu \phi)_{\underline{a}}$  are also missing since they are in the center. Therefore, these equations involve only in  $(D_\mu \phi)_{(i\vec{m})}$  components. The independent equations from (1.8) turn out to be

$$m^a C_0^\mu (D_\mu \phi)_{(i\vec{m})} = 0, \quad (2.46)$$

$$f^{ij}{}_k C_0^\mu (D_\mu \phi)_{(i\vec{m})} = 0, \quad (2.47)$$

where in the first equation (2.46) no summation with respect to  $\vec{m}$  is taken. Thus, for non-zero mode  $\vec{m} \neq \vec{0}$ , the first equation (2.46) means  $C_0^\mu (D_\mu \phi)_{(i\vec{m})} = 0$ , which also solves the second equation (2.47). For zero mode  $\vec{m} = \vec{0}$ , the first equation is trivial. If the index  $i$  for  $(D_\mu \phi)_{(i\vec{0})}$  satisfies  $f^{ij}_k = 0$  for all  $j$  and  $k$ , namely it belongs to an Abelian sub-algebra, the second equation is satisfied. Otherwise, the second one again gives the restriction  $C_0^\mu (D_\mu \phi)_{(i\vec{0})} = 0$ . Recall that  $C_0^\mu$  is nonzero only for  $\mu = \tilde{\mu}$ . We then summarize the result;

- For non zero modes,  $(D_\mu \phi)_{(i, \vec{m} \neq \vec{0})}$ , and the zero mode with indices that are not in any Abelian sub-algebra,  $(D_\mu \phi)_{(i\vec{0})}$  with  $f^{ij}_k \neq 0$  for some  $j$  and  $k$ , the constraint imposes the condition:

$$C_0^{\tilde{\mu}} (D_{\tilde{\mu}} \phi)_{(i\vec{m})} = 0, \quad (2.48)$$

that is, the covariant derivatives for the fields in tensor multiplets are suppressed in the direction of  $C_0^{\tilde{\mu}} \neq 0$ . We can therefore see that the world volume is dimensionally reduced in this  $\tilde{\mu}$  direction.

- For zero modes associated with Abelian sub-algebra, no reduction occurs. However, these modes (i.e., zero modes associated with Abelian sub-algebras) will turn out to be decoupled from the other modes that we are interested in, and therefore will not be included in the effective Lagrangians we will discuss.

With these structures in mind, we next analyze the equations of  $X^I$  and  $\Psi$ .

**$X^I$  and  $\Psi$  part** It is not difficult to see that the equations of motion (1.3) and (1.5) can be obtained by the following Lagrangians:

$$\begin{aligned} \mathcal{L}_X &= -\frac{1}{2} \langle (D_\mu X^I), (D^\mu X^I) \rangle + \frac{1}{4} \langle [C^\mu, X^I, X^J], [C_\mu, X^I, X^J] \rangle \\ &= -\frac{1}{2} (D_\mu X^I)_{(i, -\vec{m})} (D^\mu X^I)_{(i\vec{m})} + \frac{1}{4} [C^\mu, X^I, X^J]_{(i\vec{m})} [C_\mu, X^I, X^J]_{(i, -\vec{m})}, \end{aligned} \quad (2.49)$$

$$\begin{aligned} \mathcal{L}_\Psi &= \frac{i}{2} \langle \bar{\Psi}, \Gamma^\mu (D_\mu \Psi) \rangle + \frac{1}{2} \langle \bar{\Psi}, \Gamma_\nu \Gamma^I [C^\nu, X^I, \Psi] \rangle \\ &= \frac{i}{2} \bar{\Psi}_{(i, -\vec{m})} \Gamma^\mu (D_\mu \Psi)_{(i\vec{m})} + \frac{1}{2} \bar{\Psi}_{(i, -\vec{m})} \Gamma_\nu \Gamma^I [C^\nu, X^I, \Psi]_{(i\vec{m})}, \end{aligned} \quad (2.50)$$

where in each second line we have eliminated the ghost fields by using the shift symmetry.

First, we look at the covariant derivatives. By using

$$\tilde{A}_\mu^0{}_{(i\vec{m})} = -v_a \tilde{A}_\mu^a{}_{(i\vec{m})}, \quad X_0^I = \lambda_0^I, \quad X_a^I = \lambda_a^I, \quad (2.51)$$

we have

$$\begin{aligned} (D_\mu X^I)_{(i\vec{m})} &= \partial_\mu X^I_{(i\vec{m})} - \tilde{A}_\mu^{(j\vec{n})}_{(i\vec{m})} X^I_{(j\vec{n})} - \tilde{A}_\mu^{(i\vec{m})}_{(i\vec{m})} X^I_{(i\vec{m})} - \tilde{A}_\mu^0_{(i\vec{m})} X^I_0 - \tilde{A}_\mu^a_{(i\vec{m})} X^I_a \\ &= \left( \tilde{D}_\mu X^I \right)_{(i\vec{m})} - \tau_a^I (\partial^a A_\mu)_{(i\vec{m})} , \end{aligned} \quad (2.52)$$

where we have defined  $\tau_a^I \equiv \lambda_a^I - v_a \lambda_0^I$  and used (2.28)–(2.32). Here the covariant derivative  $\tilde{D}_\mu$  is defined as (A.21) in Appendix A.3.  $m^a$  can be regarded as the momentum associated with the internal direction  $y_a$ , and we replace it with the derivative with respect to  $y_a$ ,  $im^a \phi_{(i\vec{m})} = (\partial^a \phi)_{(i\vec{m})}$ . We will often employ this identification later. We now decompose the bosonic field  $X^I_{(i\vec{m})}$  into the components that are parallel to  $\tau_a^I$  and the ones perpendicular to that, by following the idea of [22]. We first introduce a projector:

$$P_J^I \equiv \delta_J^I - \tau_a^I \pi_J^a, \quad \pi_I^a \tau_b^I = \delta_b^a, \quad (2.53)$$

where the conjugate vectors  $\pi_I^a$  are defined by

$$\pi_I^a = \delta_{IJ} g^{ab} \tau_b^J, \quad (2.54)$$

where  $g_{ab} = \tau_a^I \tau_b^I$  is a metric constructed from “vielbein”  $\tau_a^I$  and is assumed to be invertible. By using this projector, we define

$$X^I_{(i\vec{m})} = P_J^I X^J_{(i\vec{m})} + \tau_a^I Y^a_{(i\vec{m})}, \quad (2.55)$$

where  $Y^a_{(i\vec{m})} \equiv \pi_J^a X^J_{(i\vec{m})}$ . Note that  $\tau_a^I$  and  $\pi_I^a$  are constant and also covariantly constant with respect to  $\tilde{D}_\mu$  since the gauge rotation in  $\tilde{D}_\mu$  is only involved in  $(i\vec{m})$  index. This fact leads to

$$(D_\mu X^I)_{(i\vec{m})} = P_J^I \left( \tilde{D}_\mu X^J \right)_{(i\vec{m})} + \tau_a^I \left( \tilde{D}_\mu Y^a - \partial^a A_\mu \right)_{(i\vec{m})}. \quad (2.56)$$

Since  $\Psi_0 = \Psi_a = 0$ , the fermion field kinetic term simply becomes

$$(D_\mu \Psi)_{(i\vec{m})} = \left( \tilde{D}_\mu \Psi \right)_{(i\vec{m})}. \quad (2.57)$$

Now we move on to the potential terms. By using the same moduli fields, we have

$$\begin{aligned} & [C^\mu, X^I, X^J]_{(i\vec{m})} \\ &= C_0^\mu \left( m^a \tau_a^I (X^J - \lambda_0^J v) - m^a \tau_a^J (X^I - \lambda_0^I v) + [X^I - \lambda_0^I v, X^J - \lambda_0^J v] \right)_{(i\vec{m})} \\ &= C_0^\mu \left( \left[ (P\tilde{X})^I, (P\tilde{X})^J \right] - i\tau_a^I \hat{\nabla}^a (P\tilde{X})^J + i\tau_a^J \hat{\nabla}^a (P\tilde{X})^I - i\tau_a^I \tau_b^J f^{ab} \right)_{(i\vec{m})}, \end{aligned} \quad (2.58)$$

where  $\tilde{X}_{(i\vec{m})}^I = X_{(i\vec{m})}^I - \lambda_0^I v_{(i\vec{m})}$  is a shifted  $X_{(i\vec{m})}^I$  and we decompose it into  $\tilde{X}_{(i\vec{m})}^I = P_J^I \tilde{X}_{(i\vec{m})}^J + \tau_a^I \tilde{Y}^a$  with  $\tilde{Y}_{(i\vec{m})}^a = Y_{(i\vec{m})}^a - \pi_J^a \lambda_0^J v_{(i\vec{m})}$ . A new covariant derivative  $\hat{\nabla}^a$  is defined by

$$\left(\hat{\nabla}^a \phi\right)_{(i\vec{m})} = \left(\partial^a \phi + i \left[\tilde{Y}^a, \phi\right]\right)_{(i\vec{m})}, \quad (2.59)$$

namely,  $\tilde{Y}_{(i\vec{m})}^a$  is treated as a new gauge field, and  $m^a$  is transformed into the derivative in  $a$  direction,  $\partial^a = i m^a$ .  $f_{(i\vec{m})}^{ab}$  is the field strength corresponding to this gauge field:

$$f_{(i\vec{m})}^{ab} = \left(\partial^a \tilde{Y}^b - \partial^b \tilde{Y}^a + i \left[\tilde{Y}^a, \tilde{Y}^b\right]\right)_{(i\vec{m})}. \quad (2.60)$$

In the fermionic part, we have the potential term including

$$\begin{aligned} [C^\nu, X^I, \Psi]_{(i\vec{m})} &= C_0^\nu m^a \tau_a^I \Psi_{(i\vec{m})} + C_0^\nu [X^I - \lambda_0^I v, \Psi]_{(i\vec{m})} \\ &= -i C_0^\nu \tau_a^I \left(\hat{\nabla}^a \Psi\right)_{(i\vec{m})} + C_0^\nu P_J^I \left[\tilde{X}^J, \Psi\right]_{(i\vec{m})}. \end{aligned} \quad (2.61)$$

Collecting all the results we can spell out the effective Lagrangian for  $X^I$  and  $\Psi$  fields. Note that since  $\tilde{D}_\mu v_{(i\vec{m})} = 0$ , we can replace  $(\tilde{D} X^I)_{(i\vec{m})}$  with  $(\tilde{D} \tilde{X}^I)_{(i\vec{m})}$ , the same for  $Y$ , and then remove all tilde from  $\tilde{X}^I$  and  $\tilde{Y}$  by redefining the fields. Therefore,  $v_{(i\vec{m})}$  plays a trivial role in  $X^I$  and  $\Psi$  part Lagrangian. Later, we will see that the gauge field equations of motion force  $v_{(i\vec{m})} = 0$ , but this change does not affect this part of the Lagrangian. Finally the effective Lagrangian of this part becomes

$$\begin{aligned} \mathcal{L}_{X+\Psi} &= -\frac{1}{2} P_{IJ} \left(\tilde{D}^\mu X^I\right)_{(i, -\vec{m})} \left(\tilde{D}_\mu X^J\right)_{(i\vec{m})} - \frac{1}{2} C^2 P_{IJ} g_{ab} \left(\hat{\nabla}^a X^I\right)_{(i, -\vec{m})} \left(\hat{\nabla}^b X^J\right)_{(i\vec{m})} \\ &\quad + \frac{1}{4} C^2 P_{IK} P_{JL} [X^I, X^J]_{(i, -\vec{m})} [X^K, X^L]_{(i\vec{m})} - \frac{1}{4} C^2 g_{ac} g_{bd} f_{(i, -\vec{m})}^{ab} f_{(i\vec{m})}^{cd} \\ &\quad - \frac{1}{2} g_{ab} \left(\tilde{D}_\mu Y^a - \partial^a A_\mu\right)_{(i, -\vec{m})} \left(\tilde{D}^\mu Y^b - \partial^b A^\mu\right)_{(i\vec{m})} \\ &\quad + \frac{i}{2} \bar{\Psi}_{(i, -\vec{m})} \left(\Gamma^\mu (\tilde{D}_\mu \Psi) - C_0^\nu \Gamma_\nu \Gamma^I \tau_a^I (\hat{\nabla}^a \Psi)\right)_{(i\vec{m})} \\ &\quad + \frac{1}{2} \bar{\Psi}_{(i, -\vec{m})} C_0^\nu \Gamma_\nu \Gamma^I P_J^I [X^J, \Psi]_{(i\vec{m})}. \end{aligned} \quad (2.62)$$

In the fermion kinetic term,  $\Gamma^\mu \partial_\mu - C_0^\nu \Gamma_\nu \Gamma^I \tau_a^I \partial^a$  is a Dirac operator in the sense of

$$(i \Gamma^\mu \partial_\mu + C_0^\mu \Gamma_\mu \Gamma^I \tau_a^I m^a) (-i \Gamma^\nu \partial_\nu + C_0^\nu \Gamma_\nu \Gamma^J \tau_b^J m^b) = \partial^\mu \partial_\mu - C_0^2 g_{ab} m^a m^b. \quad (2.63)$$

Because the formulation of [17] that we use does not have the  $SO(1, 10)$  covariance, it gives different structures of Gamma matrix between  $\tilde{D}_\mu$  and  $\hat{\nabla}^a$ .

**Gauge field  $H_{\mu\nu\rho}$  and  $\tilde{F}_{\mu\nu}$**  Finally, we consider the self-dual three-form field  $H_{\mu\nu\rho A}$  and the field strength  $\tilde{F}_{\mu\nu}{}^B{}_A$ . Because of the constraint (1.6), which is in the bracket form:

$$\tilde{F}_{\mu\nu}{}^B{}_A = i [C^\rho, H_{\mu\nu\rho}, T^B]_A, \quad (2.64)$$

$\tilde{F}_{\mu\nu}$  and  $H_{\mu\nu\rho}$  are not independent degrees of freedom when  $C^\rho \neq 0$ . Therefore we can rewrite  $H_{\mu\nu\rho}$  equation of motion (1.4) in terms of  $\tilde{F}_{\mu\nu}$ . Let us take a close look at the equation of motion for  $H_{\mu\nu\rho A}$  (1.4) that takes the form of the Bianchi identity. Thanks to the self-duality condition,  $H_{\mu\nu\rho A} = \frac{1}{3!}\epsilon_{\mu\nu\rho\sigma\tau\lambda}H^{\sigma\tau\lambda}{}_A$ , the first derivative terms can be changed by multiplying the epsilon tensor<sup>5</sup>:

$$\epsilon_{\mu\nu\rho\lambda\sigma\tau}D^{[\mu}H^{\nu\rho\lambda]}{}_A = 3!D^\mu H_{\mu\sigma\tau A}, \quad (2.65)$$

and then (1.4) can be written as the usual equation of motion:

$$D^\mu H_{\mu\nu\lambda A} + 2i [C_{[\nu}, X^I, D_{\lambda]}X^I]_A - [C_{[\nu}, \bar{\Psi}, \Gamma_{\lambda]}\Psi]_A = 0. \quad (2.66)$$

We may define a two-form “current”  $J_{\nu\lambda A}$  as

$$J_{\nu\lambda A} = 2i [C_{[\nu}, X^I, D_{\lambda]}X^I]_A - [C_{[\nu}, \bar{\Psi}, \Gamma_{\lambda]}\Psi]_A, \quad (2.67)$$

and the equation of motion is now

$$D^\mu H_{\mu\nu\lambda A} = -J_{\nu\lambda A}. \quad (2.68)$$

It should be noted that because of the three-bracket,  $J_{\mu\nu 0} = J_{\mu\nu a} = 0$  immediately follows. For  $A = \underline{0}, \underline{a}$  case, we have

$$\partial^\mu H_{\mu\nu\lambda \underline{0}} - \tilde{A}^\mu{}^B{}_{\underline{0}} H_{\mu\nu\lambda B} = -J_{\nu\lambda \underline{0}} \quad (2.69)$$

and the same for  $\underline{a}$ .  $H_{\mu\nu\lambda \underline{0}}$  appears only here among the equations of motion. Once we remove this ghost mode by gauging the shift symmetry, this equation becomes a constraint for  $J_{\nu\lambda \underline{0}}$ . As we will see, however,  $J_{\nu\lambda \underline{0}}$  will decouple from the rest of the dynamics, and then we can safely assume that this constraint is always satisfied.

Now we multiply  $D^\mu$  to (2.64). Since

$$\begin{aligned} (D^\mu [C^\rho, H_{\mu\nu\rho}, T^B])_A &= [D^\mu C^\rho, H_{\mu\nu\rho}, T^B]_A + [C^\rho, D^\mu H_{\mu\nu\rho}, T^B]_A \\ &= [C^\rho, D^\mu H_{\mu\nu\rho}, T^B]_A, \end{aligned} \quad (2.70)$$

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<sup>5</sup>See the appendix for our convention for the antisymmetrization.



we have

$$\begin{aligned}
\left(D^\mu \tilde{F}_{\mu\nu}\right)^B{}_A &= i [C^\lambda, D^\mu H_{\mu\nu\lambda}, T^B]_A \\
&= -i [C^\lambda, J_{\nu\lambda}, T^B]_A \\
&= C^\lambda_C J_{\nu\lambda D} f^{CDB}{}_A.
\end{aligned} \tag{2.71}$$

Because  $J_{\nu\lambda 0}$  and  $J_{\nu\lambda \underline{a}}$  are in the center, they do not contribute to this equation as we have anticipated. The independent equations are

$$\left(D^\mu \tilde{F}_{\mu\nu}\right)^0{}_{\underline{a}} = i m^a v_{(i, -\vec{m})} C_0^\lambda J_{\nu\lambda}{}^{(i\vec{m})}, \tag{2.72}$$

$$\left(D^\mu \tilde{F}_{\mu\nu}\right)^0{}_{(i\vec{m})} = -i m^a v_a C_0^\lambda J_{\nu\lambda}{}^{(i\vec{m})} - i [v, C_0^\lambda J_{\nu\lambda}]^{(i\vec{m})}, \tag{2.73}$$

$$\left(D^\mu \tilde{F}_{\mu\nu}\right)^a{}_{(i\vec{m})} = i m^a C_0^\lambda J_{\nu\lambda}{}^{(i\vec{m})}, \tag{2.74}$$

$$\left(D^\mu \tilde{F}_{\mu\nu}\right)^{(i\vec{m})}{}_{(i\vec{m})} = 0, \tag{2.75}$$

$$\left(D^\mu \tilde{F}_{\mu\nu}\right)^{(i\vec{m})}{}_{(j\vec{n})} = f^{ki}{}_j C_0^\lambda J_{\nu\lambda}{}^{(k, \vec{n}-\vec{m})}. \tag{2.76}$$

Therefore now  $C_0^\lambda J_{\nu\lambda}{}^{(i\vec{m})}$  plays the role of the source current for  $\tilde{F}_{\mu\nu}$ . Comparing (2.72)-(2.76) with (2.41) and (2.43), we have

$$0 = m^a \partial^\mu f_{\mu\nu a}, \tag{2.77}$$

$$0 = [v, C_0^\lambda J_{\nu\lambda}]^{(i\vec{m})}. \tag{2.78}$$

Let us examine (2.77) first. This relation holds for arbitrary  $\vec{m}$ , and then it means  $\partial^\mu f_{\mu\nu a} = 0$ . If we differentiate the effective Lagrangian (2.62) with respect to  $a_{\mu a}$ , there appears a non-zero current composed by  $X^I$  and  $\Psi$  for this equation. So the gauge fields  $a_{\mu a}$  must be regarded as the background ones, of which we do not consider the variation. Because the background fields  $a_{\mu a}$  satisfy the vacuum equation of motion, we will assume the simplest solution  $a_{\mu a} = 0$  here <sup>6</sup>. The second equation (2.78) means that  $v_{(i\vec{m})}$  must be zero apart from the zero mode  $v_{(i\vec{0})}$  <sup>7</sup>. Such zero modes associated with Abelian sub-algebra do not couple to the interaction. Moreover, as we have seen,  $v_{(i\vec{m})}$  can be absorbed into the shift of the  $X^I_{(i\vec{m})}$  and then is irrelevant for  $X^I$  and  $\Psi$  part Lagrangian. Therefore, as a solution to the equations, we can set  $v_{(i\vec{m})} = 0$ .

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<sup>6</sup>The background fields  $a_{\mu a}$  can be represented by 2-form gauge fields  $b_{\mu\nu a}$  and  $b_{\mu\nu 0}$ . This is discussed in the appendix C in detail.

<sup>7</sup>Under our assumption  $a_{\mu a} = 0$ , one can justify to set  $v_{(i\vec{m})} = 0$  also from (2.37) as well as (2.40) by an analogous discussion.

After setting  $a_{\mu a} = v_{(i\vec{m})} = 0$ , by using (2.28)-(2.32) we find that the non-zero components of  $\left(D^\mu \tilde{F}_{\mu\nu}\right)^A_B$  are

$$\left(D^\mu \tilde{F}_{\mu\nu}\right)^0_{(i\vec{m})} = im^a v_a \left(\hat{D}^\mu F_{\mu\nu}\right)_{(i\vec{m})}, \quad (2.79)$$

$$\left(D^\mu \tilde{F}_{\mu\nu}\right)^a_{(i\vec{m})} = -im^a \left(\hat{D}^\mu F_{\mu\nu}\right)_{(i\vec{m})}, \quad (2.80)$$

$$\left(D^\mu \tilde{F}_{\mu\nu}\right)^{(i\vec{m})}_{(j\vec{n})} = -f^{ki}_j \left(\hat{D}^\mu F_{\mu\nu}\right)_{(k,\vec{n}-\vec{m})} \quad (2.81)$$

where we have defined

$$F_{\mu\nu}(i\vec{m}) = (\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu])_{(i\vec{m})}. \quad (2.82)$$

Here the definition of the covariant derivative  $\hat{D}_\mu$  is given in (A.22):

$$(\hat{D}^\mu F_{\mu\nu})_{(i\vec{m})} = \partial^\mu F_{\mu\nu}(i\vec{m}) + i[A^\mu, F_{\mu\nu}]_{(i\vec{m})}. \quad (2.83)$$

Then by comparing these with (2.72)-(2.76), we have a single equation of motion for the non-vanishing gauge field:

$$\left(\hat{D}^\mu F_{\mu\nu}\right)_{(i\vec{m})} = \tilde{J}_\nu(i\vec{m}), \quad (2.84)$$

where  $\tilde{J}_\nu(i\vec{m}) = C_0^\mu J_{\mu\nu}(i\vec{m})$ .

Now one can check that the current term is derived from  $\mathcal{L}_{X+\Psi}$  in (2.62):

$$\begin{aligned} \frac{\delta \mathcal{L}_{X+\Psi}}{\delta A^\nu_{(i,-\vec{m})}} &= -g_{ab} \partial^a (\hat{D}_\nu Y^b - \partial^b A_\nu)_{(i\vec{m})} - iP_J^I \left[ X_I, \hat{D}_\nu X^J \right]_{(i\vec{m})} \\ &\quad - ig_{ab} \left[ Y^a, (\hat{D}_\nu Y^b - \partial^b A_\nu) \right]_{(i\vec{m})} + \frac{1}{2} [\bar{\Psi}, \Gamma_\nu \Psi]_{(i\vec{m})} \\ &= -\frac{1}{C^2} \tilde{J}_\nu(i\vec{m}). \end{aligned} \quad (2.85)$$

Therefore, the equation of motion can be derived from the standard Lagrangian,  $-\frac{1}{4C^2} F_{\mu\nu}(i,-\vec{m}) F^{\mu\nu}_{(i\vec{m})}$ .

**Supersymmetry** We look at the condition for supersymmetry to be preserved under a given set of the moduli. We have now,

$$X_0^I = \lambda_0^I, \quad X_a^I = \lambda_a^I, \quad C_0^{\tilde{\mu}}, H_{\mu\nu\rho 0}, H_{\mu\nu\rho a} = \text{const.}, \quad C_a^{\tilde{\mu}} = v_a C_0^{\tilde{\mu}}, \quad (2.86)$$

and the others are zero. Then the supersymmetry transformation of each component of  $\Psi$  is

$$\delta\Psi_{\underline{\alpha}} = 0, \quad \delta\Psi_{(i\vec{m})} = 0, \quad \delta\Psi_{\alpha} = \frac{1}{3!} \frac{1}{2} \Gamma^{\mu\nu\rho} H_{\mu\nu\rho\alpha}, \quad (2.87)$$

where  $\alpha = (0, a)$ . Note that  $\delta\Psi_{(i\vec{m})} = 0$  since all gauge fields  $\tilde{A}_{\mu}^B{}_A$  are set to be zero as the background. These relations mean that our choice of the moduli does not break supersymmetry if we take  $H_{\mu\nu\rho 0} = H_{\mu\nu\rho a} = 0$ . Since these components of three-form do not appear in the effective action, namely they are decoupled, then we will set them to vanish.

**Summary** In summary, we write down the effective Lagrangian in this case:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} P_{IJ} \left[ \left( \hat{D}^{\hat{\mu}} X^I \right)_{(i, -\vec{m})} \left( \hat{D}_{\hat{\mu}} X^J \right)_{(i\vec{m})} + C^2 g_{ab} \left( \hat{D}^a X^I \right)_{(i, -\vec{m})} \left( \hat{D}^b X^J \right)_{(i\vec{m})} \right] \\ & + \frac{1}{4} C^2 P_{IK} P_{JL} [X^I, X^J]_{(i, -\vec{m})} [X^K, X^L]_{(i\vec{m})} \\ & + \frac{i}{2} \bar{\Psi}_{(i, -\vec{m})} \left( \Gamma^{\hat{\mu}} (\hat{D}_{\hat{\mu}} \Psi) - C_0^{\tilde{\mu}} \Gamma_{\tilde{\mu}} \Gamma^I \tau_a^I (\hat{D}^a \Psi) \right)_{(i\vec{m})} \\ & + \frac{1}{2} \bar{\Psi}_{(i, -\vec{m})} C_0^{\tilde{\mu}} \Gamma_{\tilde{\mu}} \Gamma^I P_J^I [X^J, \Psi]_{(i\vec{m})} \\ & - \frac{1}{4C^2} \left[ F_{\hat{\mu}\hat{\nu}}{}_{(i, -\vec{m})} F_{(i\vec{m})}^{\hat{\mu}\hat{\nu}} + C^4 g_{ac} g_{bd} F_{(i, -\vec{m})}^{ab} F_{(i\vec{m})}^{cd} + 2C^2 g_{ab} \eta_{\hat{\mu}\hat{\nu}} F_{(i, -\vec{m})}^{a\hat{\mu}} F_{(i\vec{m})}^{b\hat{\nu}} \right], \end{aligned} \quad (2.88)$$

where we rewrite  $f^{ab}$  as  $F^{ab}$ , and define a dimensionless gauge field  $A_{(i\vec{m})}^a = Y_{(i\vec{m})}^a$  and covariant derivatives

$$\left( \hat{D}_{\hat{\mu}} \phi \right)_{(i\vec{m})} = \partial_{\hat{\mu}} \phi_{(i\vec{m})} + i [A_{\hat{\mu}}, \phi]_{(i\vec{m})}, \quad (2.89)$$

$$\left( \hat{D}_a \phi \right)_{(i\vec{m})} = (\partial_a \phi)_{(i\vec{m})} + i [A_a, \phi]_{(i\vec{m})}. \quad (\partial_a = i m_a) \quad (2.90)$$

$a_{\mu a}$  and  $v_{(i\vec{m})}$  have disappeared. At this stage, we have taken into account the effect of the dimensional reduction from (2.48), and  $\hat{\mu}$  and  $\hat{\nu}$  denote other directions than  $\tilde{\mu}$ , namely  $\hat{\mu} = 0, \dots, 4$  if  $\tilde{\mu} = 5$ . Note that the field strength is also restricted due to the condition (2.64).

This Lagrangian can be seen as supersymmetric Yang-Mills theory defined on  $\mathbf{R}^{1,4} \times T^d$ . Now one of the  $\mu$  direction,  $\mu = \tilde{\mu}$ , is reduced by constraint. Instead, we have added world-volume directions equipped with the metric  $g_{ab}$ . The  $C$ -field is completely a constant. There are two other parameters,  $v_a$  and  $v_{(i\vec{m})}$ , which are

proportional constants of  $C_a^{\tilde{\mu}}$  and  $C_{(i\vec{m})}^{\tilde{\mu}}$  components to  $C_0^{\tilde{\mu}}$  respectively.  $v_{(i\vec{m})}$  is set to be zero as a result of the equations of motion, and  $v_a$  is constant and is combined with the moduli parameters  $\lambda_0^I$  and  $\lambda_a^I$ , which are from  $X^I$ , to form a “vielbein”  $\tau_a^I = \lambda_a^I - v_a \lambda_0^I$ . The internal metric is given by  $g_{ab} = \tau_a^I \tau_b^I$ . An appropriate combination of  $(C_0^{\tilde{\mu}})^2$  and  $g_{ab}$  indeed gives the volume of the torus  $T^d$ .  $C_0^{\tilde{\mu}}$  appears solely as a coefficient of interaction terms, and we can view the role of  $C_0^{\tilde{\mu}}$  as a coupling constant. We will concretely see the relation in the case of  $d = 1$  later.

## 2.4 $C_0 = 0$ case

In this section, we consider the case with  $C_0^\mu = C_{(i\vec{m})}^\mu = 0$ , which solves the condition  $C_C^\mu C_D^\nu f^{CDB}{}_A = 0$ . Then this case is the case 3 of section 2.1,  $C^\mu = C_a^\mu u^a$ . As we will see, from this setup we have a different kind of action on a torus.

**$C$  fields and gauge fields** As mentioned above, we consider the case,

$$C_0^\mu = 0, \quad C_{(i\vec{m})}^\mu = 0, \quad C_a^\mu \neq 0. \quad (2.91)$$

This condition trivially solves one of the constraint equations (1.7),  $C_A^\mu C_B^\nu f^{ABC}{}_D = 0$ . We then start with the other of (1.7),  $(D_\mu C^\nu)_A = 0$ . For  $A = \underline{0}, \underline{a}$ , this becomes the ghost decoupling condition as before. Likewise the previous case, we again impose the condition that  $\partial_\nu C_{\underline{0}}^\mu = \partial_\nu C_{\underline{a}}^\mu = 0$  for decoupling of the ghost modes. For  $A = a$ , this condition means  $\partial_\mu C_a^\nu = 0$ , namely, the non-zero components of  $C_a^\mu$  have to be constant. For  $A = 0, (i\vec{m})$ , this becomes

$$\tilde{A}_{\mu \ 0}^a C_a^\nu = \tilde{A}_{\mu \ (i\vec{m})}^a C_a^\nu = 0. \quad (2.92)$$

Thus for  $a$  with non-zero  $C_a^\mu$ ,  $\tilde{A}_{\mu \ 0}^a = \tilde{A}_{\mu \ (i\vec{m})}^a = 0$ . Without tilde, this condition implies

$$A_{\mu \ (i\vec{m})(i, -\vec{m})} f^{(i\vec{m})(i, -\vec{m})0}{}_a = 0, \quad A_{\mu \ 0(i\vec{m})} f^{0(i\vec{m})a}{}_{(i\vec{m})} = 0. \quad (2.93)$$

Since the structure constant here is proportional to  $m^a$ , we have the condition,  $A_{\mu \ (i\vec{m})(i, -\vec{m})}$  for all  $\vec{m}$ , and  $A_{\mu \ 0(i\vec{m} \neq \vec{0})} = 0$ , but  $A_{\mu \ 0(i\vec{0})}$  is unconstrained. Therefore the non-vanishing gauge fields are

$$\begin{aligned} \tilde{A}_{\mu \ (i\vec{m})}^0 &= -im^a A_{\mu \ a(i\vec{m})} + f^{jk}{}_i A_{\mu \ (j\vec{n})(k, \vec{m} - \vec{n})}, \\ \tilde{A}_{\mu \ (i\vec{m})}^{(i\vec{m})} &= -im^a A_{\mu \ 0a}, \\ \tilde{A}_{\mu \ (j\vec{n})}^{(i\vec{m})} &= f^{ki}{}_j A_{\mu \ 0(k\vec{0})} \delta^{\vec{m} - \vec{n}}. \end{aligned} \quad (2.94)$$

With these gauge fields, the covariant derivatives are

$$\begin{aligned}
(D_\mu \phi)_0 &= \partial_\mu \phi_0, \quad (D_\mu \phi)_a = \partial_\mu \phi_a, \quad (D_\mu \phi)_{\underline{a}} = \partial_\mu \phi_{\underline{a}}, \\
(D_\mu \phi)_{\underline{0}} &= \partial_\mu \phi_{\underline{0}} + \tilde{A}_\mu^0{}_{(i, -\vec{m})} \phi_{(i\vec{m})}, \\
(D_\mu \phi)_{(i\vec{m})} &= \left( \tilde{D}_\mu \phi \right)_{(i\vec{m})} - \tilde{A}_\mu^0{}_{(i\vec{m})} \phi_0.
\end{aligned} \tag{2.95}$$

**$X$  and  $\Psi$  part** For the bosonic field  $X_A^I$  and the fermionic field  $\Psi_A$ , we impose the same ghost decoupling condition as before. Namely, we gauge away  $u^{\underline{0}}$  and  $u^{\underline{a}}$  components and take the conjugate components of  $X^I$  and  $\Psi$  to be moduli:

$$X_0^I = \lambda_0^I, \quad X_a^I = \lambda_a^I, \quad \Psi_0 = \Psi_a = 0, \tag{2.96}$$

where  $\lambda_0^I$  and  $\lambda_a^I$  are constant. As we will see, there will not appear  $\lambda_a^I$  in the effective equations of motion below, and then it is sufficient to consider  $\lambda_0^I$  only. We then use  $\lambda^I \equiv \lambda_0^I$  in this subsection. We can define the projector as before, which in this case takes the form:

$$P_J^I = \delta_J^I - \frac{\lambda^I \lambda_J}{\lambda^2}, \quad P_J^I \lambda^J = \lambda_I P_J^I = 0, \tag{2.97}$$

and by using this we decompose  $X_{(i\vec{m})}^I$  as  $X_{(i\vec{m})}^I = P_J^I X_{(i\vec{m})}^J + \lambda^I Y_{(i\vec{m})}$  where  $Y_{(i\vec{m})} = \lambda_J X_{(i\vec{m})}^J / \lambda^2$ . Note that since we are considering constant  $\lambda^I$ , we can choose one direction in which  $\lambda^I$  is non-vanishing, for example,  $\lambda^I = \lambda^{10} \delta^{I10}$ . For this case the projector selects the directions  $\hat{I} = 6, 7, 8, 9$ . We will revisit this point in the summary part of this subsection.

After decoupling the ghost part,  $X$  and  $\Psi$  equations of motion are given through the generic Lagrangian (2.49) and (2.50) again. We substitute our ansatz into them, and then obtain

$$\begin{aligned}
\mathcal{L}_{X+\Psi} &= -\frac{1}{2} P_{IJ} \left( \left( \tilde{D}_\mu X^I \right)_{(i, -\vec{m})} \left( \tilde{D}^\mu X^J \right)_{(i\vec{m})} + \lambda^2 \tilde{g}_{ab} (\partial^a X^I)_{(i, -\vec{m})} (\partial^b X^J)_{(i\vec{m})} \right) \\
&\quad - \frac{\lambda^2}{2} \left( \left( \tilde{D}_\mu Y \right)_{(i, -\vec{m})} - \tilde{A}_\mu^0{}_{(i, -\vec{m})} \right) \left( \left( \tilde{D}^\mu Y \right)_{(i\vec{m})} - \tilde{A}^{\mu 0}{}_{(i\vec{m})} \right) \\
&\quad + \frac{i}{2} \bar{\Psi}_{(i, -\vec{m})} \Gamma^\mu \left( \tilde{D}_\mu \Psi + \lambda^I \Gamma^I C_{\mu a} \partial^a \Psi \right)_{(i\vec{m})},
\end{aligned} \tag{2.98}$$

where  $\tilde{g}_{ab} \equiv C_a^\mu C_{\mu b}$  and again  $\partial^a = im^a$ . Therefore in this case we do not have potential terms for  $X^I$  and  $\Psi$ .

**Gauge field part** Also in this case, the gauge field equation of motion is given through (1.4) and (1.6) by (2.71):

$$\left(D^\mu \tilde{F}_{\mu\nu}\right)^B{}_A = C_C^\lambda J_{\nu\lambda}{}_D f^{CDB}{}_A, \quad (2.99)$$

where the “current”  $J_{\nu\lambda}{}_A$  is again given by (2.67):

$$J_{\nu\lambda}{}_A = 2i \left[ C_{[\nu}, X^I, D_{\lambda]} X^I \right]_A - \left[ C_{[\nu}, \bar{\Psi}, \Gamma_{\lambda]} \Psi \right]_A. \quad (2.100)$$

As before, only  $J_{\nu\lambda}{}_{(i\vec{m})}$  components contributes to the equation of motion. The ghost part of  $H_{\mu\nu\rho}{}_{\underline{0}}$  and  $H_{\mu\nu\rho}{}_{\underline{a}}$  are taken care of in the same way as before. Since now only  $C_a^\mu$  is non-zero, the only non-trivial equation of motion is

$$\begin{aligned} \left(D^\mu \tilde{F}_{\mu\nu}\right)^0{}_{(i\vec{m})} &= -im^b C_b^\lambda J_{\nu\lambda}{}_{(i\vec{m})} \\ &= \lambda^2 \tilde{g}_{ab} m^a m^b \left( (\tilde{D}_\nu Y)_{(i\vec{m})} - \tilde{A}_\nu^0{}_{(i\vec{m})} \right), \end{aligned} \quad (2.101)$$

and the other components are  $\left(D^\mu \tilde{F}_{\mu\nu}\right)^A{}_B = 0$ .

By defining  $A_{\mu\,0(k\vec{0})} = A_{\mu\,k}^{(0)}$  and  $A_{\mu\,0a} = -a_{\mu\,a}$ , the field strength can be written down as

$$\tilde{F}_{\mu\nu}{}^{(i\vec{m})}{}_{(j\vec{n})} = -f^{ki}{}_j F_{\mu\nu\,k}^{(0)} \delta_{\vec{n}}^{\vec{m}}, \quad \tilde{F}_{\mu\nu}{}^{(i\vec{m})}{}_{(i\vec{m})} = -im^a f_{\mu\nu\,a}, \quad (2.102)$$

where  $F_{\mu\nu\,k}^{(0)} = 2\partial_{[\mu} A_{\nu]\,k}^{(0)} - 2f^{ij}{}_k A_{[\mu\,i}^{(0)} A_{\nu]\,j}^{(0)}$ ,  $f_{\mu\nu\,a} = 2\partial_{[\mu} a_{\nu]\,a}$ .  $\tilde{F}_{\mu\nu}^0{}_{(i\vec{m})}$  is also non-zero and depends on all the gauge fields  $A_{\mu\,k}^{(0)}$ ,  $a_{\mu\,a}$  and  $\tilde{A}_\mu^0{}_{(i\vec{m})}$ . The rest of the gauge field strengths, including  $\tilde{F}_{\mu\nu}^0{}_{\underline{a}}$  and  $\tilde{F}_{\mu\nu}^a{}_{(i\vec{m})}$ , all vanish. We start with the equations of motion with  $(i\vec{m})(j\vec{n})$  index and  $(i\vec{m})(i\vec{m})$  index. As mentioned above, there is no source term for these components and the equations of motion are

$$-f^{ki}{}_j \left( \partial^\mu F_{\mu\nu\,k}^{(0)} - 2f^{st}{}_k A_{[s}^{(0)\,\mu} F_{t]\,\mu\nu}^{(0)} \right) = 0, \quad -im^a \partial^\mu f_{\mu\nu\,a} = 0. \quad (2.103)$$

On the other hand, if we differentiate the effective Lagrangian (2.98) for  $X^I$  and  $\Psi$  after the gauge fields  $A_{\mu\,k}^{(0)}$  and  $a_{\mu\,a}$ , it is easy to see that we have non-zero currents consisting of  $X^I$  and  $\Psi$ . So these gauge fields have to be regarded as backgrounds, so that we do not consider the variations of these fields in the action. Since the background gauge fields satisfy the equations of motion without sources (2.103), we will assume the simplest solution  $A_{\mu\,k}^{(0)} = a_{\mu\,a} = 0$ . Note that this choice makes the covariant derivative  $\tilde{D}_\mu$  appearing in the action (2.98) the ordinary one  $\partial_\mu$ . Finally, the nontrivial equation of motion for the gauge field becomes

$$\partial^\mu \tilde{F}_{\mu\nu}^0{}_{(i\vec{m})} = \lambda^2 \tilde{g}_{ab} m^a m^b \left( \partial_\nu Y_{(i\vec{m})} - \tilde{A}_\nu^0{}_{(i\vec{m})} \right)$$

$$= \tilde{g}_{ab} m^a m^b \frac{\delta \mathcal{L}_{X+\Psi}}{\delta \tilde{A}^{\nu 0}_{(i, -\vec{m})}}, \quad (2.104)$$

and  $\tilde{F}_{\mu\nu}^0_{(i\vec{m})} = -\partial_\mu \tilde{A}_\nu^0_{(i\vec{m})} + \partial_\nu \tilde{A}_\mu^0_{(i\vec{m})}$  is an Abelian field strength. Therefore we just have copies of the Abelian gauge field.

Note that the gauge field  $\tilde{A}_\mu^0_{(i\vec{m})}$  appears either in the combination  $\partial_\nu Y_{(i\vec{m})} - \tilde{A}_\nu^0_{(i\vec{m})}$  or in the field strength  $\tilde{F}_{\mu\nu}^0_{(i\vec{m})}$ . Let us consider the residual gauge transformation, with the gauge parameter  $\tilde{\Lambda}^0_{(i\vec{m})}$ , of the fields. It is easy to see that  $P_J^I X_{(i\vec{m})}^J$  and  $\Psi_{(i\vec{m})}$  do not transform under this gauge transformation, while  $Y_{(i\vec{m})}$  transforms as  $Y_{(i\vec{m})} \rightarrow Y_{(i\vec{m})} + \tilde{\Lambda}^0_{(i\vec{m})}$ . We have set the background field condition,  $A_{\mu k}^{(0)} = a_{\mu a} = 0$ . Next we write the term involving  $Y_{(i\vec{m})}$  in the effective Lagrangian (2.98) in the Fourier transformed form:

$$- \frac{\lambda^2}{2} \int_0^{2\pi} \frac{d^d y}{(2\pi)^d} (\partial_\mu Y_i(y) - A_{\mu i}(y)) (\partial^\mu Y_i(y) - A^\mu_i(y)), \quad (2.105)$$

where  $Y_i(y) = \sum_{\vec{m}} Y_{(i\vec{m})} e^{i\vec{m} \cdot \vec{y}}$  and  $A_{\mu i}(y) = \sum_{\vec{m}} \tilde{A}_\mu^0_{(i\vec{m})} e^{i\vec{m} \cdot \vec{y}}$ , and the suffix  $i$  is just the label for Abelian fields of which we now have a number of copies. Then this term turns out to be written in the form of the  $U(1)$  complex Higgs kinetic term as

$$- \frac{1}{2} \int_0^{2\pi} \frac{d^d y}{(2\pi)^d} (\mathcal{D}_\mu \varphi_i)^\dagger (\mathcal{D}^\mu \varphi_i), \quad (2.106)$$

where  $\varphi_i(y) = \sqrt{|\lambda|^2} e^{iY_i(y)}$ , and the covariant derivative is  $\mathcal{D}_\mu \varphi_i = \partial_\mu \varphi_i - i A_{\mu i} \varphi_i$ , where the index  $i$  is not summed over. Note that  $Y_i(y)$  is dimensionless since  $\lambda^I Y_{(i\vec{m})}$  has the same mass dimension as  $\langle X_0^I \rangle = \lambda^I$ . The fluctuation with respect to the magnitude  $|\varphi_i|$  comes from the fluctuation of  $X_0^I$ . This fluctuation is suppressed as a result of the ghost decoupling discussed in section 2.2. So there is no fluctuation along the absolute value.

The part (2.106) has the  $U(1)$  gauge symmetries as

$$\varphi_i \rightarrow e^{i\Lambda_i(y)} \varphi_i, \quad (2.107)$$

with the Fourier transformed gauge parameters  $\Lambda_i(y)$ . Now the action (2.98) is the expansion of the (2.106) around the following vacuum expectation values (VEVs)<sup>8</sup>,

$$\langle \varphi_i \rangle = \sqrt{|\lambda|^2}, \quad \langle Y_i \rangle = 0. \quad (2.108)$$

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<sup>8</sup>Though we only deal with the classical equations of motion in this paper, we abuse the term VEV to refer to constant solutions to the equations of motion, around which we derive new equations of motion for the dynamical fields. This would not cause any confusion.

These VEVs break the  $U(1)$  gauge transformations (2.107). Then we can regard  $Y_i(y)$  as the Goldstone modes along the broken  $U(1)$  direction. Since  $Y_i(y)$  are the Goldstone modes, like the usual Higgs mechanism, we can absorb these modes by redefining the gauge fields  $A_i(y)$ . In terms of the original Fourier basis, we define

$$w_{\mu(i\vec{m})} = -\tilde{A}_{\mu(i\vec{m})}^0 + \partial_{\mu} Y_{(i\vec{m})}. \quad (2.109)$$

Then the field strength is now written in terms of  $w_{\mu(i\vec{m})}$ :

$$\begin{aligned} \tilde{F}_{\mu\nu(i\vec{m})}^0 &= -\left(\partial_{\mu}\tilde{A}_{\nu(i\vec{m})}^0 - \partial_{\nu}\tilde{A}_{\mu(i\vec{m})}^0\right) \\ &= (\partial_{\mu}w_{\nu} - \partial_{\nu}w_{\mu})_{(i\vec{m})} \\ &= W_{\mu\nu(i\vec{m})}. \end{aligned} \quad (2.110)$$

The  $w_{\mu(i\vec{m})}$  are the massive gauge bosons which already absorb the Goldstone modes  $Y_{(i\vec{m})}$ , and the equations of motion for  $Y_{(i\vec{m})}$  and  $\tilde{A}_{\mu(i\vec{m})}^0$  part can be obtained from the following W-part Lagrangian:

$$\mathcal{L}_W = -\frac{\lambda^2}{2}\bar{g}_{ab}m^am^bw_{(i,-\vec{m})}^{\mu}w_{\mu(i\vec{m})} - \frac{1}{4}W_{(i,-\vec{m})}^{\mu\nu}W_{\mu\nu(i\vec{m})}. \quad (2.111)$$

The first term can be seen as the mass term for W-bosons produced by the  $U(1)$  breaking. The number of the independent polarization for each  $w_{\mu(i\vec{m})}$  is  $6 - d - 2 + 1 = 5 - d$ <sup>9</sup>. Here  $-d$  is due to the constraint (1.8),  $-2$  is for elimination of the temporal and the longitudinal modes, and  $+1$  is from absorption of  $Y$  boson.

Next let us consider the geometrical meaning of this Higgs mechanism in terms of the target space description. This Higgs mechanism eliminates the one of the transverse directions from the action, and then can be considered as the dimensional reduction of M-theory to type IIA string theory. We identify  $\sqrt{|\lambda|^2}$  as the radius of the circle, and the phase  $Y_{(i\vec{m})}$  of  $\varphi$  as the coordinate along the M-circle. The VEV  $\langle\varphi\rangle = \sqrt{|\lambda|^2}$  as well as  $\langle Y_{(i\vec{m})}\rangle = 0$  represents the position of the 5-brane in the compactified direction, and the 5-brane breaks the shift symmetry along the compactified direction. Now because of the projection,  $Y_{(i\vec{m})}$  enjoys not only the global shift symmetry but also the gauged one, namely  $U(1)$  symmetry. As we will see in section 2.5, the gauge field  $\tilde{A}_{\mu(i\vec{m})}^0$  can be viewed as the background graviphoton field arising from the compactification. Therefore, it is natural that these gauge

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<sup>9</sup>After the projection, the number of the transverse bosons is 4 and then the total number of bosonic degrees of freedom is now  $9 - d$ , while the fermionic one is 8. Therefore, the effective Lagrangian may not be supersymmetric except for  $d = 1$  case. In the case of  $d = 1$ , the supersymmetry is similar to D4-brane's because of the dimensional reduction.



fields, corresponding to the local reparametrization on the circle, absorb the Goldstone modes and become massive. As a result, we have an effective Lagrangian of five-brane in string theory.

Since there is a constraint:

$$m^a C_a^\mu (D_\mu \phi)_{(i\vec{m})} = 0, \quad (2.112)$$

one might wonder if the theory is capable of realizing the target space whose dimensions less than ten dimensions. But the theory remains ten-dimensional even under the constraint (2.112), as we now observe. At the first sight, (2.112) prohibits the covariant derivative along the direction of  $C^\mu$ . The number of the target space dimensions  $\mathbf{R}^{1,9}$  and the world-volume dimensions  $\mathbf{R}^{1,5}$  are reduced as  $\mathbf{R}^{1,9} \rightarrow \mathbf{R}^{1,9-d}$  and  $\mathbf{R}^{1,5} \rightarrow \mathbf{R}^{1,5-d}$  respectively. But the reduced directions are recovered by the KK-momentum  $C_a^\mu \partial^a$ , and thus the actual target space and the world-volume are  $\mathbf{R}^{1,9-d} \times T^d$  and  $\mathbf{R}^{1,5-d} \times T^d$  respectively. So the theory remains to be a 5-brane effective theory of a (1+9)-dimensional superstring theory, irrespective of how many  $C_a^\mu$  we have turned on.

**Summary** We have the effective Lagrangian:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} P_{\hat{I}\hat{J}} \left( \left( \partial_\mu X^{\hat{I}} \right)_{(i,-\vec{m})} \left( \partial^\mu X^{\hat{J}} \right)_{(i\vec{m})} + \lambda^2 \tilde{g}_{ab} \left( \partial^a X^{\hat{I}} \right)_{(i,-\vec{m})} \left( \partial^b X^{\hat{J}} \right)_{(i\vec{m})} \right) \\ & + \frac{i}{2} \bar{\Psi}_{(i,-\vec{m})} \Gamma^\mu \left( \partial_\mu \Psi + \lambda^I \Gamma^I C_{\mu a} \partial^a \Psi \right)_{(i\vec{m})} \\ & - \frac{\lambda^2}{2} \tilde{g}_{ab} m^a m^b w_{(i,-\vec{m})}^\mu w_{\mu (i\vec{m})} - \frac{1}{4} W_{(i,-\vec{m})}^{\mu\nu} W_{\mu\nu (i\vec{m})}, \end{aligned} \quad (2.113)$$

where  $\tilde{g}_{ab} = C_a^\mu C_{\mu b}$ .  $\lambda_a^I = X_a^I$  do not show up here and can then be set to zero. For simplicity,  $\lambda^I = X_0^I$  is chosen as  $\lambda^I = \lambda^{10} \delta^{I10}$  and then  $P_{\hat{I}\hat{J}}$  is the projection onto  $\hat{I} = 6, 7, 8, 9$  plane. The world volume is  $\mathbf{R}^{1,5-d} \times T^d$ . Indices  $\mu, \nu$  label  $\mathbf{R}^{1,5-d}$  directions, and  $a, b$  label  $T^d$  directions. This Lagrangian might be able to couple to background gauge fields  $A_{\mu i}^{(0)}$  and  $a_{\mu a}$  by replacing the derivative with covariant derivatives, but now the background fields are turned off.  $w_\mu$  field is defined by

$$w_{\mu (i\vec{m})} = -\tilde{A}_{\mu (i\vec{m})}^0 + \partial_\mu Y_{(i\vec{m})}, \quad (2.114)$$

and thus this Lagrangian is a gauge fixed Lagrangian with massive Abelian vector bosons. All the fields have the Kaluza-Klein mass term whose mass is determined by  $\lambda^2 \tilde{g}_{ab}^{10}$ .

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<sup>10</sup> The Higgs mechanism in this section is a kind of Stückelberg mechanism [30] (for a review, see [31]). It has been known that non-Abelian extension of Stückelberg mechanism has the problems

The theory is regarded as the effective theory of Abelian 5-branes in a superstring theory, which is similar to the D5-brane or NS5-brane effective theory in type IIB superstring theory. Moreover we have seen that the compactification to string theory occurs along the direction transverse to the 5-brane world-volume, and such compactification usually gives NS5-branes in type IIA string theory. However this 5-brane action should be recognized as a type IIB NS5-brane action derived from the type IIA NS5-brane by T-duality, since the KK-momentum shows up along the 5-brane world-volume directions. This theory is interesting since it captures the history of the 5-branes generated through the M-theory compactification of M5-branes and T-duality of type IIA NS5-branes.

## 2.5 Vanishing $C$ field case

In this subsection, we consider the simplest case, i.e., all  $C_A^\mu$  vanish. This is the case 4 in section 2.1. In this case,  $\tilde{F}_{\mu\nu}{}^B{}_A = 0$  because of (1.6); that is, the auxiliary gauge field  $\tilde{A}_\mu{}^B{}_A$  is a pure gauge, but still couples to the other fields through the covariant derivatives. The equations of motion are reduced to

$$D^2 X_A^I = 0, \quad D_{[\mu} H_{\nu\lambda\rho]}{}_A = 0, \quad \Gamma^\mu (D_\mu \Psi)_A = 0. \quad (2.115)$$

Apart from the covariant derivative, these are just the equations of motion for Abelian  $(2, 0)$  tensor multiplets in six dimensions. Therefore, we may assume the Lagrangian *à la* Pasti-Sorokin-Tonin (PST [23]):

$$\mathcal{L} = -\frac{1}{2} \langle (D^\mu X^I), (D_\mu X^I) \rangle + \frac{i}{2} \langle \bar{\Psi}, \Gamma^\mu (D_\mu \Psi) \rangle + \frac{1}{4} \langle H_{\mu\nu}^*, (H^{*\mu\nu} - H^{\mu\nu}) \rangle, \quad (2.116)$$

where

$$H_{\mu\nu}{}_A = \frac{\partial^\rho a}{\sqrt{\partial_\mu a \partial^\mu a}} H_{\mu\nu\rho}{}_A, \quad H_{\mu\nu}^*{}_A = \frac{\partial^\rho a}{\sqrt{\partial_\mu a \partial^\mu a}} \frac{\epsilon_{\mu\nu\rho\lambda\sigma\tau}}{3!} H^{\lambda\sigma\tau}{}_A, \quad (2.117)$$

and now the three-form can be written in terms of a two-form potential  $b_{\mu\nu}{}_A$  as  $H_{\mu\nu\rho}{}_A = 3D_{[\mu} b_{\nu\rho]}{}_A$ , thanks to the usual Bianchi identity and  $\tilde{F}_{\mu\nu}{}^B{}_A = 0$ . Note that the three-form  $H_{\mu\nu\rho}{}_A$  is assumed to be not self-dual.  $a$  is an auxiliary scalar field of

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of renormalizability and unitarity in 4-dimension. In this paper, the effective Lagrangian is reduced to an Abelian system, regardless of the 3-algebraic structure in the formulation, and then there will not be such problems.

PST action and is singlet under the three algebraic transformation, and with it the effective Lagrangian (2.116) enjoys the following local symmetries:

$$\begin{aligned}
\text{(I): } & \delta_I a = 0, \quad \delta_I b_{\mu\nu A} = 2D_{[\mu}\xi_{\nu] A}, \\
\text{(II): } & \delta_{II} a = 0, \quad \delta_{II} b_{\mu\nu A} = 2\partial_{[\nu}a\eta_{\mu] A}, \\
\text{(III): } & \delta_{III} a = \zeta, \quad \delta_{III} b_{\mu\nu A} = \frac{\zeta}{\sqrt{(\partial a)^2}} (H_{\mu\nu A}^* - H_{\mu\nu A}). \quad (2.118)
\end{aligned}$$

The first transformation  $\delta_I$  agrees with the usual gauge symmetry of the  $b_{\mu\nu A}$  field because of  $[D_\mu, D_\nu] = 0$ . The second and the third symmetries are characteristic for the PST formalism and are important for the three-form to be on-shell self-dual. Therefore, the effective action can reproduce the equations of motion (2.115), including the linear self-duality condition.

Likewise the previous cases, we gauge the shift symmetry and then gauge away the ghost modes,  $X_0^I$  etc. In this case, we have the Lagrangian description for the three-form field strength  $H_{\mu\nu\rho A}$  and then can perform a similar treatment to  $X^I$  and  $\Psi$ . Namely, we can gauge the translation symmetry  $b_{\mu\nu \underline{a}} \rightarrow b_{\mu\nu \underline{a}} + \zeta_{\mu\nu \underline{a}}$ , where  $\underline{a} = (\underline{0}, \underline{a})$ , by promoting  $\zeta_{\mu\nu \underline{a}}$  to be local and introducing the corresponding three-form gauge field  $G_{\mu\nu\rho \underline{a}}$  as  $H_{\mu\nu\rho \underline{a}} \rightarrow H_{\mu\nu\rho \underline{a}} - G_{\mu\nu\rho \underline{a}}$ . It should be noted that the gauge transformation of  $G$  is therefore  $\delta G_{\mu\nu\rho \underline{a}} = 3D_{[\mu}\zeta_{\nu\rho] \underline{a}}$ . Then we can eliminate  $H_{\mu\nu\rho \underline{a}}$  by the gauge symmetry and the equations of motion of  $G$  gives the condition  $H_{\mu\nu\rho 0} = H_{\mu\nu\rho a} = 0$ . In this way, the fields of  $u^0$  and  $u^a$  components are again being moduli:

$$X_0^I = \lambda_0^I, \quad X_a^I = \lambda_a^I, \quad \Psi_0 = \Psi_a = H_{\mu\nu\rho 0} = H_{\mu\nu\rho a} = 0. \quad (2.119)$$

We introduce the indices  $\alpha, \beta$  to represent  $(0, a = 1 \dots d)$  indices collectively. Then  $\lambda_\alpha^I$  are  $5 \times (d+1)$  matrices, and we define  $(d+1) \times 5$  matrices  $\pi_I^\alpha$  such that

$$\lambda_\alpha^I \pi_I^\beta = \delta_\alpha^\beta. \quad (2.120)$$

Such  $\pi_I^\alpha$  can exist when  $d \leq 4$ , and we simply assume their existence in this discussion. Finally, we define the projector  $P_J^I = \delta_J^I - \lambda_\alpha^I \pi_J^\alpha$  and introduce the decomposition  $X_{(i\vec{m})}^I = P_J^I X_{(i\vec{m})}^J + \lambda_\alpha^I Y_{(i\vec{m})}^\alpha$  as before. Here  $Y_{(i\vec{m})}^\alpha = \pi_J^\alpha X_{(i\vec{m})}^J$ . Then the effective action (2.116) becomes

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{2} P_{IJ} \left( \tilde{D}^\mu X^I \right)_{(i, -\vec{m})} \left( \tilde{D}_\mu X^J \right)_{(i\vec{m})} \\
& -\frac{1}{2} \lambda_\alpha^I \lambda_\beta^I \left( \tilde{D}^\mu Y_{(i, -\vec{m})}^\alpha - \tilde{A}^{\mu\alpha}_{(i, -\vec{m})} \right) \left( \tilde{D}_\mu Y_{(i\vec{m})}^\beta - \tilde{A}_{\mu (i\vec{m})}^\beta \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{i}{2} \bar{\Psi}_{(i, -\vec{m})} \Gamma^\mu \left( \tilde{D}_\mu \Psi \right)_{(i\vec{m})} \\
& + \frac{1}{4} H_{\mu\nu}^*{}_{(i, -\vec{m})} (H^{*\mu\nu} - H^{\mu\nu})_{(i\vec{m})} .
\end{aligned} \tag{2.121}$$

**Interpretation of the effective action** Let us consider the brane interpretation of our effective action (2.121). First note that by setting all  $\lambda_\alpha^I = 0$ , the effective action is, apart from the flat connection gauge field  $\tilde{A}_\mu{}^B{}_A$ , nothing but the PST Lagrangian:

$$\mathcal{L}_{\text{PST}} = - \left( \sqrt{-\det(g_{\mu\nu} + iH_{\mu\nu}^*)} + \frac{1}{4} H_{\mu\nu}^* H^{\mu\nu} \right) , \tag{2.122}$$

$$\begin{aligned}
g_{\mu\nu} &= \partial_\mu X^M \partial_\nu X^N G_{MN} , \\
H_{\mu\nu} &= \frac{\partial^\rho a}{\sqrt{(\partial a)^2}} H_{\mu\nu\rho} , \quad H_{\mu\nu}^* = \frac{\partial^\rho a}{\sqrt{(\partial a)^2}} \frac{\epsilon_{\mu\nu\rho\lambda\sigma\tau}}{3!} H^{\lambda\sigma\tau} ,
\end{aligned} \tag{2.123}$$

expanded up to the quadratic order in the flat metric with the static gauge:

$$G_{MN} = \begin{pmatrix} \eta_{\mu\nu} & \\ & \delta_{IJ} \end{pmatrix} , \quad X^M = (x^\mu, X^I) , \tag{2.124}$$

where  $M, N = (\mu, I)$ . For the current purpose, it is sufficient to consider only the bosonic part of the action. To compare it to the case with non-zero  $\lambda_\alpha^I$ , we consider the following Kaluza-Klein compactification ansatz,

$$G_{MN} = \begin{pmatrix} \eta_{\mu\nu} + g_{\alpha\beta} A_\mu{}^\alpha A_\nu{}^\beta - g_{\beta\gamma} A_\mu{}^\gamma & 0 \\ -g_{\alpha\gamma} A_\nu{}^\gamma & g_{\alpha\beta} & 0 \\ 0 & 0 & \delta_{\hat{I}\hat{J}} \end{pmatrix} , \tag{2.125}$$

with the static gauge  $X^M = (x^\mu, Y^\alpha, X^{\hat{I}})$ . With this ansatz, (2.122) becomes, up to the quadratic order in the physical fields:

$$\begin{aligned}
\mathcal{L}_{\text{PST}} &= - \frac{1}{2} \left( \partial^\mu X^{\hat{I}} \right) \left( \partial_\mu X^{\hat{I}} \right) - \frac{1}{2} g_{\alpha\beta} \eta^{\mu\nu} (\partial_\mu Y^\alpha - A_\mu{}^\alpha) (\partial_\nu Y^\beta - A_\nu{}^\beta) \\
&+ \frac{1}{4} H_{\mu\nu}^* (H^{*\mu\nu} - H^{\mu\nu}) ,
\end{aligned} \tag{2.126}$$

where we have dropped an uninteresting constant term.

We compare the resulting Lagrangian with our effective Lagrangian (2.121). Now because of the projector, some of  $I, J$  directions are eliminated. Therefore the un-eliminated index  $I, J$  can be identified with  $\hat{I}, \hat{J}$  here. The projected scalars  $Y_{(i\vec{m})}^a$  are identified with the directions in which the Kaluza-Klein reduction has

been performed. The gauge fields  $\tilde{A}_{\mu (i\vec{m})}^{\alpha}$  are regarded as the graviphoton gauge fields from the reduction. Because of  $C_A^{\mu} = 0$ , there is no relation between  $\tilde{A}_{\mu (i\vec{m})}^{\alpha}$  and  $H_{\mu\nu\rho (i\vec{m})}$  in this case. It is consistent to the fact that  $\tilde{A}_{\mu (i\vec{m})}^{\alpha}$  is identified with an external graviphoton. It should be noted that since  $\tilde{A}_{\mu (i\vec{m})}^{\alpha}$  is pure gauge, the corresponding graviphoton field should also be trivial one. The fermions and the three-form field strength are naturally understood. The metric  $g_{\alpha\beta}$  should be identified with  $\lambda_{\alpha}^I \lambda_{\beta}^I$  in (2.121), and then the target space is regarded as  $\mathbf{R}^{1,9-d} \times M_{d+1}$  where  $M_{d+1}$  is a  $d + 1$  dimensional manifold with the metric  $g_{\alpha\beta} = \lambda_{\alpha}^I \lambda_{\beta}^I$ . In the effective action, the index  $(i\vec{m})$  shows that the fields are bunch of Abelian 5-brane fields which are interacting only through the covariant derivative. However, as seen, the field strength of the connection vanishes, and then these copies of the Abelian fields are very loosely communicating each other. Thus we have a (almost) trivially interacting Abelian fields on a five-brane in  $\mathbf{R}^{1,9-d} \times M_{d+1}$ . This is a (non-Abelian) generalization of the second order NS5-brane Lagrangian discussed in [24]. In our case, the dimensional reduction is done to more than one direction.

### 3 Five-brane actions and duality relations

As it has already been shown, the effective actions derived from the equations of motion of non-Abelian  $(2, 0)$  tensor multiplets in six dimensions correspond to various brane effective actions on some torus. Since we would like to understand the starting equations of motion as a kind of effective description of multiple M5-branes, one question naturally arises: do these effective actions respect the symmetries of M-theory, especially string duality?

To answer this question, we analyze the effective actions in the case of  $d = 1$ , namely the label for the Lorentzian generator  $a$  takes only one value  $a = 1$ . This setup leads to different kinds of 5-branes, and we will investigate the relation between their effective actions.

#### 3.1 D5-branes and NS5-branes in type IIB theory

For the case of  $d = 1$ , we have various five-brane actions. We will start with 5-branes with  $C_0^{\mu}, C_a^{\mu} \neq 0$  case, as in section 2.3. The effective Lagrangian in this case is written as

$$\mathcal{L}_5 = -\frac{1}{2} \left[ \left( \hat{D}^{\mu} X^{\hat{I}} \right)_{(i, -\vec{m})} \left( \hat{D}_{\mu} X^{\hat{I}} \right)_{(i\vec{m})} + C^2 \tau^2 \left( \hat{D}_{\hat{a}} X^{\hat{I}} \right)_{(i, -\vec{m})} \left( \hat{D}_{\hat{a}} X^{\hat{I}} \right)_{(i\vec{m})} \right]$$

$$\begin{aligned}
& + \frac{1}{4} C^2 \left[ X^{\hat{I}}, X^{\hat{J}} \right]_{(i, -\vec{m})} \left[ X^{\hat{I}}, X^{\hat{J}} \right]_{(i\vec{m})} \\
& + \frac{i}{2} \bar{\Psi}_{(i, -\vec{m})} \left( \Gamma^\mu (\hat{D}_\mu \Psi) - C \Gamma_5 \Gamma^6 \tau (\hat{D}_{\hat{a}} \Psi) \right)_{(i\vec{m})} \\
& + \frac{1}{2} \bar{\Psi}_{(i, -\vec{m})} C \Gamma_5 \Gamma^{\hat{I}} \left[ X^{\hat{I}}, \Psi \right]_{(i\vec{m})} \\
& - \frac{1}{4C^2} \left[ F_{\mu\nu (i, -\vec{m})} F_{(i\vec{m})}^{\mu\nu} + 2C^2 \tau^2 \eta_{\mu\nu} F_{(i, -\vec{m})}^{\hat{a}\mu} F_{(i\vec{m})}^{\hat{a}\nu} \right] , \tag{3.1}
\end{aligned}$$

where  $C_0^{\tilde{\mu}=5} = C$ . Since we now have only  $a = 1$ , we can make a rotation to set  $\tau_1^I = \tau_1^6 = \tau \delta^{I6}$ . Subsequently, it is easy to see that the projector  $P_J^I = \delta_J^{\hat{I}}$  where  $\hat{I}, \hat{J} = 7, 8, 9, 10$ . Namely,  $P_J^I$  provides a projection to a plane perpendicular to the 6 direction. So far, we have only one internal coordinate  $y$ , corresponding to the direction which is denoted as  $\hat{a}$  hereafter. The actual direction will be specified soon below.

This theory is compactified on a circle, whose radius is at first considered to be constant. The Fourier modes are expanded with the basis  $e^{iy^m}$ , and then  $y$  becomes dimensionless. Since the combination  $C\tau$  always has mass dimension equal to 1, we can redefine the dimensionless coordinates  $y$  and dimensionless gauge field  $A_{\hat{a}} = Y$  as

$$y \rightarrow C\tau y, \quad A_{\hat{a}} \rightarrow (C\tau)^{-1} A_{\hat{a}}. \tag{3.2}$$

Then the periodicity of  $y$  is equal to  $y \sim y + 2\pi/C\tau$ , and the radius of the circle is identified with  $R = 1/(C\tau)$ .

**D5-brane** Since we have started with the equations of motion, the overall factor of the Lagrangian cannot be fixed. Consequently, we assume a pre-factor which is equivalent to the  $D5$ -brane tension  $T_5 = (2\pi)^{-5} \ell_s^{-6} g_s^{-1}$ , and compare our effective action with the  $D5$ -brane action. By introducing  $\hat{\mu}, \hat{\nu} = 0, 1, 2, 3, 4, \hat{a}$ , we write the effective action as follows:

$$\begin{aligned}
S_{eff} = & -T_5 \int d^5x \sum_m \left( \frac{1}{2} \left( \hat{D}_{\hat{\mu}} X^{\hat{I}} \right)_{(i, -\vec{m})} \left( \hat{D}_{\hat{\mu}} X^{\hat{I}} \right)_{(i\vec{m})} + C^2 \tau^2 m^2 X_{(i, -\vec{m})}^{\hat{I}} X_{(i\vec{m})}^{\hat{I}} \right. \\
& \left. - \frac{1}{4} C^2 \left[ X^{\hat{I}}, X^{\hat{J}} \right]_{(i, -\vec{m})} \left[ X^{\hat{I}}, X^{\hat{J}} \right]_{(i\vec{m})} + \frac{1}{4C^2} F_{(i, -\vec{m})}^{\hat{\mu}\hat{\nu}} F_{\hat{\mu}\hat{\nu} (i\vec{m})} + (\text{fermions}) \right) . \tag{3.3}
\end{aligned}$$

The relevant part of the action of D5-brane on  $S^1$  of radius  $R$ , which can be derived as the Yang-Mills limit of the Dirac-Born-Infeld (DBI) action, is

$$S_{D5} = -T_5 \int d^5x \sum_m \left( \frac{1}{2} \left( \hat{D}_{\hat{\mu}} X^I \right)_{(i, -\vec{m})} \left( \hat{D}_{\hat{\mu}} X^I \right)_{(i\vec{m})} + \frac{m^2}{R^2} X_{(i, -\vec{m})}^I X_{(i\vec{m})}^I \right. \\ \left. - \frac{1}{4(2\pi\alpha')^2} [X^I, X^J]_{(i, -\vec{m})} [X^I, X^J]_{(i\vec{m})} + \frac{(2\pi\alpha')^2}{4} F_{(i, -\vec{m})}^{\hat{\mu}\hat{\nu}} F_{\hat{\mu}\hat{\nu}}(i\vec{m}) + \dots \right). \quad (3.4)$$

Thus we can identify

$$C^2 = \frac{1}{(2\pi\alpha')^2}, \quad R = \frac{1}{C\tau}, \quad \tau \propto \frac{\alpha'}{R} = R_{IIA}. \quad (3.5)$$

It should be noted that the parameter  $\tau$  is related to the radius of the T-duality circle  $R_{IIA}$  in the type IIA theory description. Based on the identification of  $\tau$ , we have KK-momentum modes along the  $\tau = \tau_1^6$  direction, and have set  $\hat{a} = 6$  here. As a result, the world-volume extends on this 6 direction as well. This can be understood as T-duality in the specific direction. On the other hand, because of  $C_0^5$ , the reduced direction 5 is understood as the M-theory direction, and we could think the 5 direction as a circle of radius  $R_M = g_s \ell_s$ . For this case, we can analyze the preserved supersymmetry, and the theory has indeed non-chiral  $\mathcal{N} = (1, 1)$  supersymmetry. See Appendix B for details. Therefore, the action (3.3) is the D5-brane action of type IIB string theory.

One may wonder why this identification is so different from the one which Lambert and Papageorgakis took in [17], where  $C$  is identified as the coupling constant, and more suitable for the interpretation of the novel Higgs mechanism. To understand the difference, we choose another coefficient to describe the effective action:

$$S_5 = -\frac{1}{C^2} \int d^6x \, \text{tr} \left[ \frac{1}{2} \left( \hat{D}_{\hat{\mu}} X^I \right)^2 - \frac{1}{4} [X^I, X^J]^2 + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \dots \right], \quad (3.6)$$

where  $d^6x$  includes  $dy$  (of length dimension 1) and  $\text{tr}$  denotes the summation over index  $i$ . Since  $F_{\mu\nu}$  has a mass dimension of 2,  $X^I$  and  $C$  thus have a mass dimension of 1 and  $-1$ , respectively. The overall factor should be identified with the Yang-Mills coupling constant in six dimensions, and also be proportional to the string coupling as follows:

$$C^2 = \left( g_{\text{YM}}^{(6)} \right)^2 = g_s \alpha'. \quad (3.7)$$

Upon reduction to five dimensions, this might be related to the identification in [17]. We however preferred our previous choice because it is more convenient to figure out another choice of the parameters for the description of NS5-branes.

**NS5-brane** The previous identification of the  $S^1$  direction is natural, since the covariant derivative along the internal directions has always been in accordance with  $\tau_a^I$  as  $\tau_a^I \hat{D}^a$ . Therefore, we can interpret  $\tau_a^I$  as a vielbein to transform internal directions to directions transverse to the brane. However, for  $d = 1$  case,  $\vec{m}$  has only one component, i.e., it is a number. Furthermore, since  $m$  always appears with  $C_0^5$ , we can interpret  $C_0^5 i m = C_0^5 \partial_y$  as a derivative along the 5 direction, with  $C_0^5$  as a vielbein. To see what this re-interpretation leads, we will set the 5 direction as an internal direction,  $\hat{a} = 5$ .

Once we have chosen the 5 direction as the  $S^1$  direction in which Kaluza-Klein momentum is defined and T-duality will be taken, we can consider the other direction, which is specified by  $\tau = \tau_1^6$ , as the compactification direction for M-theory. Subsequently, we first obtain NS5-brane in type IIA theory through the reduction of the 6 direction, which is transverse to the world-volume of 5-brane. Further compactification on  $S^1$  in the 5 direction leads to T-duality, and finally we get NS5-brane in type IIB theory. Let us look at how it works. The radius of  $S^1$  along direction 5 is given by the expression  $R = (C\tau)^{-1}$ . On the other hand,  $\tau$  has the dimension of length, and is indeed related to the magnitude of the vanishing direction,  $\tau = \tau_1^6 = X_1^6 - v_1 X_0^6$ . Therefore, it is natural that we identify it with the radius of the M-circle,  $|\tau| = c_1 \hat{g}_s \ell_s$ , where a proportional constant  $c_1$  is inserted for the sake of generality. It should be noted that we use a different label for the string coupling,  $\hat{g}_s$ . In the following subsection, this identification is more justified in an effective theory which is related to the current model in an Abelian limit. These relations lead to

$$C = \frac{1}{c_1 \hat{g}_s \ell_s R} = \frac{1}{c_1 c_2} \frac{1}{\hat{g}_s \ell_s^2}, \quad (3.8)$$

where  $R = c_2 \ell_s$  with a constant of order probably larger than 1 is not involved in the string coupling. By plugging this in (3.3), the result can be identified with (3.4) with the following replacements:

$$g_s \rightarrow \hat{g}_s = g_s^{-1}, \quad \alpha' \rightarrow \hat{g}_s \alpha'. \quad (3.9)$$

These are the standard S-duality relations in type IIB string theory. It should be noted that this change also alters the tension in front of the action as

$$T_5 = \frac{1}{g_s (2\pi)^5 \ell_s^6} \rightarrow \frac{1}{\hat{g}_s^2 (2\pi)^5 \ell_s^6}, \quad (3.10)$$

which is exactly the NS5-brane tension. Therefore, by switching the interpretation, we have NS5-brane effective action in type IIB string theory. It should be noted



that the S-duality transformation takes place together with the conversion of the interpretation of  $\tau$ . In the case of D5-branes,  $\tau$  is regarded as the size of the circle of T-duality,  $\tau = R_{IIA}$ . On the other hand, in the case of NS5-branes,  $\tau$  is regarded as the size of the M-circle,  $\tau = R_M$ .

Here we have identified the internal direction  $\hat{a}$  with the 5 or the 6 directions. We then have D5-branes for  $\hat{a} = 5$  and NS5-branes for  $\hat{a} = 6$  respectively. We take either  $C_0^5$  or  $\tau^6$  as the vielbein used to transform the internal direction  $y$  into the physical direction, and the direction  $\hat{a}$  is determined by which of these we choose. This dimensional reduction might be understood as a torus compactification, and one of them is decompactified by means of KK momentum. Changing the direction of the expansion by KK-modes corresponds to the flip of the direction of the torus. Therefore, we can see a similarity to the “9-11” flip realization of S-duality of type IIB string theory.

### 3.2 Other five-branes and their relations

Next we consider the case of  $C_0^\mu = 0, C_a^\mu \neq 0$  with  $d = 1$ , in section 2.4. Since  $d = 1$ , we have only a single non-zero  $C_a^\mu$ , and, by a rotation, we can set it to  $C_1^5$ , while other components to vanish. In addition, to be able to compare the resulting action with that in the previous subsection, we define  $\tilde{A}_{\mu(i\vec{m})}^0$  as  $-(\partial^5 A_\mu)_{(i\vec{m})}$  with  $\partial^5 = im$ , motivated by the identification in section 2.3. It should be noted that this is not removing tilde by taking the structure constant off, but just a redefinition of the gauge field. Furthermore if we identify  $Y_{(i\vec{m})}$  with  $-A_{5(i\vec{m})}$ , the field strength  $F_{5\mu(i\vec{m})}$  can be defined as follows:

$$\partial_\mu Y_{(i\vec{m})} - \tilde{A}_{\mu(i\vec{m})}^0 = -\partial_\mu A_{5(i\vec{m})} + (\partial_5 A_\mu)_{(i\vec{m})} = F_{5\mu(i\vec{m})}. \quad (3.11)$$

Through the redefinition of the gauge field, we also have  $\tilde{F}_{\mu\nu(i\vec{m})}^0 = (\partial_5 (\partial_\mu A_\nu - \partial_\nu A_\mu))_{(i\vec{m})} \equiv (\partial_5 F_{\mu\nu})_{(i\vec{m})}$ . Thus, from the first line of (2.104), we obtain the following gauge field equation:

$$\partial^\mu F_{\mu\nu} + \lambda^2 \tilde{g}_{55} \partial^5 F_{5\nu} = 0. \quad (3.12)$$

The effective Lagrangian is the free part of the Yang-Mills type Lagrangian for 5-branes:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} P_{IJ} \left( (\partial_\mu X^I)_{(i, -\vec{m})} (\partial^\mu X^J)_{(i\vec{m})} + \lambda^2 (C_1^5)^2 (\partial^5 X^I)_{(i, -\vec{m})} (\partial^5 X^J)_{(i\vec{m})} \right) \\ & + \frac{i}{2} \bar{\Psi}_{(i, -\vec{m})} (\Gamma^\mu \partial_\mu \Psi + \lambda^I \Gamma^5 \Gamma^I C_1^5 \partial_5 \Psi)_{(i\vec{m})} \end{aligned}$$

$$-\frac{1}{4} \left( F_{(i,-\vec{m})}^{\mu\nu} F_{\mu\nu (i\vec{m})} + 2\lambda^2 (C_1^5)^2 F_{(i,-\vec{m})}^{5\mu} F_{5\mu (i\vec{m})} \right). \quad (3.13)$$

It is easy to see that this Lagrangian can be obtained by the free field limit  $C_0^5 \rightarrow 0$  with the combination  $v_1 C_0^5 = C_1^5$  fixed in the generic 5-brane Lagrangian (3.1). It should be noted that we also need to rescale  $F_{\mu\nu (i\vec{m})}$  by  $C_0^5$ , and this limit also corresponds to the weak field limit. Therefore it is natural that we end up with missing covariant derivatives. In this case,  $\lambda$  indeed provides the size of the M-circle, as discussed in section 2.4. Since this case is related to the NS5-brane case in the previous subsection with  $\tau = v_1 \lambda$ , the identification developed earlier is also justified.

In section 2.5, we find that the case of  $C_A^\mu = 0$  results in a second order PST type effective Lagrangian. This is a free IIA NS5-brane without compactification along the world-volume directions. The corresponding situation here is that we only have  $\lambda_0^I$  and  $\lambda_1^J$  since  $d = 1$ . The  $T^2$  compactification corresponds to taking only  $\lambda_0^{10}$  and  $\lambda_1^6$  to be nonzero. Let us now compare this to the  $C_a \neq 0$  free IIB NS5-brane wrapping on a circle in section 2.4, the Lagrangian is (3.13). Their relation can be understood as a T-dual relation.  $C_a^5$  gives the size of compactification, while the circle shrinks as  $C_a^5$  becomes gradually smaller. At  $C_a^5 = 0$  the T-dual circle has infinite size, and the  $C_a^5 \rightarrow 0$  IIB NS5-brane relates to the  $C_A = 0$  IIA NS5-brane at this point.

Finally, we consider the following limit: In the Lagrangian (3.1), we take  $\tau \rightarrow 0$  with  $C_0^5$  fixed. Since in this limit,  $v_a$  disappears from the Lagrangian and so does  $C_1^5$ . Therefore, in the limit we have the  $D4$ -branes effective Lagrangian which is the same as the one studied in [17]. Note that since the radius of the circle is given by  $R = 1/\sqrt{C^2 \tau^2}$ , this is the decompactification limit in the IIB side, and then we have double-dimensionally reduced  $D4$ -branes in IIA here. We also remark that if the limit  $v_1 \rightarrow 0$  in (3.1), the Lagrangian remains essentially the same, but replacing  $\tau^6$  with  $\lambda_1^6$ . This describes  $D5$ -branes in IIB, and by comparing to NS5-branes in IIB (3.13) we see that the role of the moduli is just switched, namely  $(C_0^5, \lambda_1^6)$  for  $D5$  and  $(C_1^5, \lambda_0^6)$  for NS5. This also resembles the “9-11” flip realization of S-duality in type IIB string theory.

## 4 Conclusion and Discussion

In this paper, we examined the equations of motion proposed by Lambert and Papageorgakis for non-Abelian  $(2, 0)$  tensor multiplets in six dimensions [17]. Some of these equations are regarded as constraint equations for non-dynamical fields,  $C_A^\mu$ .

We consider various cases where different components of  $C_A^\mu$  take non-zero values. With respect to the cases, which components of  $C_A^\mu$  are zero, we derive various equations of motion for  $p$ -branes on a backgrounds consisting of a flat space and a torus. We found that, in order to maintain a non-Abelian interaction after taking into account possible constraints, the  $u^0$  component of the  $C$  field has to be non-zero. Otherwise, we have equations of motion of Abelian fields which are loosely bound. Of particular interest is the case where  $d = 1$ , with  $d$  being the dimension of the torus. In this case, we have a circle and 5-branes. For the generic  $C$  case, we have Yang-Mills type actions, and by identifying the appropriate parameters, we have the description of either a  $D5$ -brane or  $NS5$ -brane. In the case of type IIB string theory, the S-duality of 5-branes can be interpreted as the interchange of roles between two moduli fields, which specify the compactified circles of M-theory. We therefore observe that the formulation of non-Abelian tensor multiplets seems to be compatible with the expectation from the string duality. In contrast, if only  $C_a \neq 0$ , we obtain the Lagrangian for Abelian 5-branes; this corresponds to the free field limit of the previous case. We also found that the case of zero  $C$  corresponds to the second order PST-type 5-brane action. It is worth noting that the almost free Abelian theory still includes covariant derivatives with flat connections. The second order PST-type action can also be considered to correspond to a limit of the previous two cases and be compatible with the expected T-duality.

In the following paragraphs, we will discuss the findings of this work in detail and propose possible directions for future research.

**Non-Abelian multiple 5-branes with (2,0) supersymmetry?** Through the present study, we found multiple 5-branes with (1,1)-type and (2,0)-type world-volume supersymmetries. More specifically, 5-branes with (1,1) supersymmetry were described for the case of  $C_0 \neq 0$  by means of ordinary non-Abelian SYM. In the case of the type IIB string theory, they were described as D/NS 5-branes. On the other hand, in the case of the M-theory and the type IIA string theory, 5-branes are characterized by (2,0)-type world-volume supersymmetry. These are identified to be the 5-branes corresponding to the  $C_A = 0$  case. These (2,0)-type 5-branes are also characterized by non-Abelian interactions under gauge fields. However, as we have shown in previous sections, this does not mean that our results are satisfactory. Despite the “non-Abelian extension” using the 3-algebra, these (2,0)-type multiple 5-branes have trivial non-Abelian interactions.

Indeed, there is a no-go theorem proposed in papers [32], claiming that it is impossible to obtain the desired non-Abelian extension of the Abelian chiral 2-form

theory in six dimensions using local deformation terms <sup>11</sup>. In the light of this, we think one possible way of overcoming the problem through the introduction of non-local deformation of the supersymmetry algebra (1.1) <sup>12</sup>, or alternatively, we may consider a quantization of the Nambu bracket [34] <sup>13</sup> to introduce an interesting interaction. This problem still remains open.

We also pose the question about the  $N^3$  entropy scaling law of an  $N$  M5-brane system [36]. We recall that the (truncated-)Nambu-Poisson algebra has been used to provide a 3-algebraic explanation for the  $N^{3/2}$  entropy scaling law of an  $N$  coincident M2-brane system [16]. This algebra may provide the means of explaining the M5-branes entropy.

**5D MSYM** Finally, we give a remark that our findings are related to recent research on the 5D MSYM theory [25][26]. The authors of these papers discuss the relation between 5D MSYM and 6D  $(2,0)$  superconformal field theory on  $S^1$ . More specifically, KK-modes of the  $(2,0)$  theory, associated with  $S^1$  compactification, can be explained as solitonic states in 5D MSYM.

In contrast, in our study, the  $(2,0)$  theory was employed to describe  $S^1$  compactifications of the 6D theory in a number of special cases. The KK-momentum was provided by 3-algebra and independent of the degrees of freedom of 5D SYM. Therefore, to explain KK-modes in our analysis as the solitonic states in 5D MSYM, some non-trivial relations in addition to the field equations (1.3)-(1.8) are necessary.

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<sup>11</sup>This theorem was claimed to the theory of chiral 2-form without other fields. On the other hand, the authors of [12] found that there is no S-matrix of the (weakly-)interacting  $(2,0)$  tensor multiplet provided that we assume it to have a Lagrangian description.

<sup>12</sup> Some works explored to introduce non-locality by considering loop space variables [33].

<sup>13</sup> In [35], the authors propose a criterion which any valid quantization of the generalized Takhtajan theory should satisfy. It is very interesting to check that the quantization in [34] satisfies the criterion or not.

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## A Summary for notation

### A.1 A Lorentzian 3-algebra

We use the loop extension of the Lorentzian 3-algebra which is also used in [22], but with a slightly different notation. The generators are collectively denoted as

$$T^A = \{T^{(i\vec{m})}, u^0, u^a, u^{\underline{0}}, u^{\underline{a}}\}, \quad (\text{A.1})$$

namely,  $A$  denotes all the indices collectively,  $A = \{(i\vec{m}), 0, a, \underline{0}, \underline{a}\}$ . We sometimes use the index  $\alpha = (0, a)$  and  $\underline{\alpha} = (\underline{0}, \underline{a})$ . We assume that our 3-algebra is equipped with the gauge invariant metric  $g_{AB}$

$$g_{(i\vec{m})(j\vec{n})} = \delta_{ij} \delta_{\vec{m}+\vec{n}}, \quad g_{0\underline{0}} = 1, \quad g_{a\underline{b}} = \delta_{ab}, \quad (\text{A.2})$$

and the other components are zero. Here we conventionally define the generator  $u_a$  representing the center element  $u^{\underline{a}}$  obtained by the metric as

$$u_a \equiv g_{a\underline{b}} u^{\underline{b}} = u^{\underline{a}}. \quad (\text{A.3})$$

Therefore the inner products are

$$\begin{aligned} \langle T^{(i\vec{m})}, T^{(j\vec{n})} \rangle &= \delta^{ij} \delta^{\vec{m}+\vec{n}}, & \langle u^a, u_b \rangle &= \langle u^a, u^{\underline{c}} g_{\underline{c}b} \rangle = g^{a\underline{c}} g_{\underline{c}b} = \delta^a_b, \\ \langle u^0, u_0 \rangle &= \langle u^0, u^{\underline{0}} \rangle = 1, \end{aligned} \quad (\text{A.4})$$

and otherwise zero.

The structure constant of the 3-algebra is essentially the same as the one in [22]. The nonzero components of the totally antisymmetric structure constant are

$$f^{0a(i\vec{m})(j\vec{n})} = -im^a \delta^{ij} \delta^{\vec{m}+\vec{n}}, \quad f^{0(i\vec{m})(j\vec{n})(k\vec{\ell})} = f^{ijk} \delta^{\vec{m}+\vec{n}+\vec{\ell}}, \quad (\text{A.5})$$

which satisfy the usual fundamental identity,

$$f^{ABC}{}_F f^{FDE}{}_G + f^{ABD}{}_F f^{CFE}{}_G + f^{ABE}{}_F f^{CDF}{}_G = f^{CDE}{}_F f^{ABF}{}_G. \quad (\text{A.6})$$

The generators  $u^{\underline{0}}$  and  $u^{\underline{a}}$  are center, in the sense that they do not appear as the upper index of the structure constant, namely, when they are put inside the three-bracket defined below, the result is zero. On the other hand, the generators  $u^0$  and

$u^a$  are not in the lower indices of the structure constant, and then they do not show up as a result of the three-bracket operation.

We sometimes use the three-bracket expression,

$$[u^0, u^a, u^b] = 0, \quad (\text{A.7})$$

$$[u^0, u^a, T^{(i\vec{m})}] = m^a T^{(i\vec{m})}, \quad (\text{A.8})$$

$$[u^0, T^{(i\vec{m})}, T^{(j\vec{n})}] = m^a \delta^{ij} \delta^{\vec{m}+\vec{n}} u_a + i f^{ij}_k T^{(k, \vec{m}+\vec{n})}, \quad (\text{A.9})$$

$$[u^a, T^{(i\vec{m})}, T^{(j\vec{n})}] = -m^a \delta^{ij} \delta^{\vec{m}+\vec{n}} u^{\underline{0}}, \quad (\text{A.10})$$

$$[T^{(i\vec{m})}, T^{(j\vec{n})}, T^{(k\vec{\ell})}] = -i f^{ijk} \delta^{\vec{m}+\vec{n}+\vec{\ell}} u^{\underline{0}}, \quad (\text{A.11})$$

where  $[T^A, T^B, T^C] = i f^{ABC}_D T^D$ .

The indices are normally used in the following conventions:

- $I, J, \dots$ : the transverse directions of five branes ( $I, J = 6, \dots, 10$ ).
- $\mu, \nu, \dots$ : world-volume directions of five branes ( $\mu, \nu = 0, \dots, 5$ ).
- $d$ : The number of the Lorentzian generators(-1):  $a, \underline{a} = 1, \dots, d$ .
- $y_a$ : The coordinates for the torus  $T^d$ . The Fourier basis along the torus is  $e^{im^a y_a}$  and  $\partial^a = \frac{\partial}{\partial y_a}$ . We will regard the  $T^{(i\vec{m})}$  components of the fields,  $\phi_{(i\vec{m})}$ , as the Fourier components of the field in  $y$  coordinates,

$$\phi(x, y) = \sum_{\vec{m}} \phi_{(i\vec{m})}(x) e^{i\vec{m} \cdot \vec{y}}, \quad (\text{A.12})$$

and also use the expression  $im^a \phi_{(i\vec{m})} = (\partial^a \phi)_{(i\vec{m})}$ .

## A.2 Covariant derivative, field strength

Our convention for the covariant derivative and the field strength is the same as in [17].

- Covariant derivative:  $(D_\mu \phi)_A = \partial_\mu \phi_A - \tilde{A}_\mu^B{}_A \phi_B$
- Field strength:  $\tilde{F}_{\mu\nu}^B{}_A = -\partial_\mu \tilde{A}_\nu^B{}_A + \partial_\nu \tilde{A}_\mu^B{}_A + \tilde{A}_\mu^C{}_A \tilde{A}_\nu^B{}_C - \tilde{A}_\nu^C{}_A \tilde{A}_\mu^B{}_C$
- Covariant derivative of field strength:  

$$\left( D^\mu \tilde{F}_{\mu\nu} \right)^B{}_A = \partial^\mu \tilde{F}_{\mu\nu}^B{}_A + \tilde{A}^{\mu B}{}_C \tilde{F}_{\mu\nu}^C{}_A - \tilde{A}^{\mu C}{}_A \tilde{F}_{\mu\nu}^B{}_C$$

### A.3 Assumption for the gauge field

We assume that the gauge fields  $\tilde{A}_\mu$  are accompanied with the structure constant of 3-algebra,

$$\tilde{A}_{\mu A}^B = A_{\mu CD} f^{CDB}_A, \quad (\text{A.13})$$

which guarantees that it acts on the three-bracket as a derivation. Due to the limited form of the structure constant, we have

$$\tilde{A}_{\mu \underline{0}}^0 = \tilde{A}_{\mu \underline{b}}^a = \tilde{A}_{\mu 0}^A = \tilde{A}_{\mu a}^A = \tilde{A}_{\mu A}^0 = \tilde{A}_{\mu A}^a = 0. \quad (\text{A.14})$$

The rest can be written in terms of the gauge field without tilde,

$$\begin{aligned} \tilde{A}_{\mu (i\vec{m})}^0 &= f^{jk}_i A_{\mu (j\vec{n})(k, \vec{m}-\vec{n})} - im^a A_{\mu a(i\vec{m})} \\ &= f^{jk}_i A_{\mu (j\vec{n})(k, \vec{m}-\vec{n})} - (\partial^a A_{\mu a})(i\vec{m}), \end{aligned} \quad (\text{A.15})$$

$$\tilde{A}_{\mu (i\vec{m})}^a = im^a A_{\mu 0(i\vec{m})} = (\partial^a A_{\mu 0})(i\vec{m}), \quad (\text{A.16})$$

$$\tilde{A}_{\mu \underline{a}}^0 = -im^a A_{\mu (i\vec{m})(i, -\vec{m})}, \quad (\text{A.17})$$

$$\tilde{A}_{\mu (i\vec{m})}^{(i\vec{m})} = -im^a A_{\mu 0a}, \quad (\text{A.18})$$

$$\tilde{A}_{\mu (j\vec{n})}^{(i\vec{m})} = f^{ki}_j A_{\mu 0(k, \vec{n}-\vec{m})}. \quad (\text{A.19})$$

Since some components of the gauge field are zero by 3-algebra as summarized above, the covariant derivatives take the following form before further restriction,

$$(D_\mu \phi)_{(i\vec{m})} = \left( \tilde{D}_\mu \phi \right)_{(i\vec{m})} - \tilde{A}_{\mu (i\vec{m})}^0 \phi_0 - \tilde{A}_{\mu (i\vec{m})}^a \phi_a, \quad (\text{A.20})$$

$$\left( \tilde{D}_\mu \phi \right)_{(i\vec{m})} = \partial_\mu \phi_{(i\vec{m})} - \tilde{A}_{\mu (i\vec{m})}^{(i\vec{m})} \phi_{(i\vec{m})} - \tilde{A}_{\mu (i\vec{m})}^{(j\vec{n})} \phi_{(j\vec{n})}, \quad (\text{A.21})$$

$$\left( \hat{D}_\mu \phi \right)_{(i\vec{m})} = \partial_\mu \phi_{(i\vec{m})} - \tilde{A}_{\mu (i\vec{m})}^{(j\vec{n})} \phi_{(j\vec{n})} = \partial_\mu \phi_{(i\vec{m})} + i[A_\mu, \phi]_{(i\vec{m})} \quad (\text{A.22})$$

where  $\phi_A$  denotes collectively the physical fields  $X_A^I$ ,  $\Psi_A$  and  $H_{\mu\nu\rho A}$ .

### A.4 Anti-symmetrization

We use the following convention for the totally anti-symmetric combination of the indices:

$$A^{[\mu} B^{\nu]} = \frac{1}{2} (A^\mu B^\nu - A^\nu B^\mu), \quad (\text{A.23})$$

$$A^{[\mu} B^\nu C^{\rho]} = \frac{1}{3!} (A^\mu B^\nu C^\rho - A^\nu B^\mu C^\rho + (4 \text{ other terms})), \quad (\text{A.24})$$

namely, in general,

$$A_1^{[\mu_1} A_2^{\mu_2} \cdots A_n^{\mu_n]} = \frac{1}{n!} (A_1^{\mu_1} A_2^{\mu_2} \cdots A_n^{\mu_n} - A_1^{\mu_2} A_2^{\mu_1} \cdots A_n^{\mu_n} + \cdots) , \quad (\text{A.25})$$

where  $\cdots$  denotes the all possible combination of the indices with sign.

So, for example,

$$D_{[\mu} H_{\nu\rho\sigma]} = \frac{1}{4} (D_\mu H_{\nu\rho\sigma} - D_\nu H_{\rho\sigma\mu} + D_\rho H_{\sigma\mu\nu} - D_\sigma H_{\mu\nu\rho}) . \quad (\text{A.26})$$

## B $\mathcal{N} = (1, 1)$ supersymmetry

We discuss how  $\mathcal{N} = (1, 1)$  supersymmetry of D5-brane in section 3 is realized from  $\mathcal{N} = (2, 0)$  supersymmetric set up.

We use the same convention as in [17] for Gamma matrices. Supersymmetry is parametrized by 16-component spinor  $\epsilon$ . The chirality of  $\epsilon$  is described by

$$\Gamma^{012345} \epsilon = \epsilon. \quad (\text{B.1})$$

In section 3, the 5th direction of the D5-brane world volume is reduced by the consequence of (1.8), and other world volume direction, say  $a = 6$ , is created by KK momentum. Thus the natural chirality operator for this 5-brane is  $\Gamma^{012346}$ , instead of  $\Gamma^{012345}$ . Because of  $\{\Gamma^{012345}, \Gamma^{012346}\} = 0$ ,  $\epsilon$  in (B.1) contains both chirality states of  $\Gamma^{012346}$ . If we take a basis of the spinor

$$\epsilon = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix}, \quad \Gamma^{012346} \epsilon_\pm = \pm \epsilon_\pm, \quad (\text{B.2})$$

$\Gamma^{012345}$  can be written by a  $16 \times 16$  matrix  $\gamma^{012345}$

$$\Gamma^{012345} = \begin{pmatrix} 0 & \gamma^{012345} \\ \gamma^{012345} & 0 \end{pmatrix}. \quad (\text{B.3})$$

The supersymmetric condition (B.1) gives

$$\gamma^{012345} \epsilon_\pm = \epsilon_\pm. \quad (\text{B.4})$$

Since  $\gamma^{012345}$  is invertible, the numbers of the components of  $\epsilon_\pm$  are the same. They describe the vector-like  $(1, 1)$  supersymmetry in six dimensions, as expected for D5-brane.



## C The relationship between $a^\mu$ and the 2-form gauge field $b_{\mu\nu}$

From (1.4), it is easy to see that for  $H_{\mu\nu\rho\alpha}$  the equations of motion are the usual Bianchi identity  $\partial_{[\mu}H_{\nu\rho\sigma]\alpha} = 0$ , and then these 3-form field strength can be represented by the 2-form fields  $b_{\mu\nu\alpha}$ ,

$$H_{\nu\rho\sigma\alpha} = \partial_\nu b_{\rho\sigma\alpha} + \partial_\rho b_{\sigma\nu\alpha} + \partial_\sigma b_{\nu\rho\alpha}. \quad (\text{C.1})$$

From (1.6) and the relation (A.18), one can see that the field strength  $f_{\mu\nu a}$  of the gauge field  $a_{\mu a} \equiv A_{\mu a0}$  is represented by the 3-form field strength as

$$m^b f_{\mu\nu b} = -m^a (C_0^\rho H_{\mu\nu\rho a} - C_a^\rho H_{\mu\nu\rho 0}), \quad (\text{C.2})$$

where  $f_{\mu\nu a} = (\partial_\mu a_{\nu a} - \partial_\nu a_{\mu a})$ . The constraint (1.8) causes the dimensional reduction of the three-form field, and therefore we take the two-form potential to obey the relation,

$$C_0^\rho \partial_\rho b_{\mu\nu a} - C_a^\rho \partial_\rho b_{\mu\nu 0} = 0, \quad (\text{C.3})$$

as well as the one-form gauge transformation as we will see soon. We can then identify

$$a_{\mu a} = -C_0^\nu b_{\mu\nu a} + C_a^\nu b_{\mu\nu 0}. \quad (\text{C.4})$$

Under the dimensional reduction, the  $U(1)$  gauge transformation of the 2-form gauge fields  $b_{\mu\nu\alpha}$ ,

$$b_{\mu\nu\alpha} \rightarrow b_{\mu\nu\alpha} + \delta b_{\mu\nu\alpha} = b_{\mu\nu\alpha} + \partial_\mu \Lambda'_{\nu\alpha} - \partial_\nu \Lambda'_{\mu\alpha}, \quad (\text{C.5})$$

is naturally identified with the  $U(1)$  gauge transformation of  $a_{\mu a} \rightarrow a_{\mu a} + \partial_\mu \Lambda_a$  through the identification (C.4)<sup>14</sup>. Through the (C.4), the  $U(1)$  gauge transformation of the 2-form gauge field generate the gauge transformation of the 1-form gauge field as

$$\begin{aligned} \delta a_{\mu a} &= -C_0^\nu \delta b_{\mu\nu a} + C_a^\nu \delta b_{\mu\nu 0} \\ &= -2C_0^\nu \partial_{[\mu} \Lambda'_{\nu]a} + 2C_a^\nu \partial_{[\mu} \Lambda'_{\nu]0} \\ &= -C_0^\nu \partial_\mu \Lambda'_{\nu a} + C_a^\nu \partial_\mu \Lambda'_{\nu 0} \end{aligned} \quad (\text{C.7})$$

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<sup>14</sup>The  $U(1)$  gauge transformation is generated by the gauge transformation parameter with the Lie 3-algebra

$$\tilde{\Lambda}^{(i\vec{m})}_{(i\vec{m})} = -im_a \Lambda_{0a} = im_a \Lambda_a. \quad (\text{C.6})$$

due to the dimensional reduction  $C_0^\nu \partial_\nu \Lambda'_{\mu a} - C_a^\nu \partial_\nu \Lambda'_{\mu 0} = 0$ . Finally the  $U(1)$  gauge transformation of the 1-form gauge field can be unified to the  $U(1)$  gauge transformation of the 2-form gauge field through the identification of the parameter

$$\Lambda_a = -C_0^\nu \Lambda'_{\nu a} + C_a^\nu \Lambda'_{\nu 0}. \quad (\text{C.8})$$

This relation is interesting since, at the beginning, the 3-algebra gauge transformation and the two-form gauge field transformation are the different things, but now they are unified<sup>15</sup>. If it also worked in the non-Abelian part, this mechanism would give a significant suggestion for the non-Abelian generalization of two-form gauge field (or maybe higher form even). Unfortunately, it seems that this unification can be confirmed only in this Abelian sector.

Since the three-form field strength  $H_{\mu\nu\rho\alpha}$  is self-dual, the Bianchi identity implies that it satisfies the usual equation of motion without sources,

$$\partial^\mu H_{\mu\nu\rho\alpha} = 0. \quad (\text{C.9})$$

This suggests that  $m^a \partial^\mu f_{\mu\nu a} = 0$  for arbitrary  $\vec{m}$ , and then  $\partial^\mu f_{\mu\nu a} = 0$ . However the rest of the dynamical fields,  $X_{(i\vec{m})}^I$ ,  $\Psi_{(i\vec{m})}$  and so on, couple to  $a_{\mu a}$  through the covariant derivative, and then they will generate the source term for this field strength. In order for this equation of motion to hold, we need to regard  $a_{\mu a}$  as the background field and we do not take the variation with respect to  $a_{\mu a}$  in the other part of the Lagrangian.

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<sup>15</sup>Only a part of 2-form gauge transformation is related to 1-form one. 2-form gauge transformations that are not related to  $C^\mu$  directions are invisible by the 1-form  $a_\mu$ .

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